Introduction

A Lévy process is essentially a stochastic process with stationary and independent increments.

The basic theory was developed, principally by Paul Lévy in the 1930s. In the past 20 years there has been a renaissance of interest and a plethora of books, articles and conferences. Why?

There are both theoretical and practical reasons.

Noise. Lévy processes are a good model of “noise” in random dynamical systems.

\[ \text{Input} + \text{Noise} = \text{Output} \]

Attempts to describe this differentially leads to stochastic calculus. A large class of Markov processes can be built as solutions of stochastic differential equations (SDEs) driven by Lévy noise. Lévy driven stochastic partial differential equations (SPDEs) are currently being studied with some intensity.

Robust structure. Most applications utilise Lévy processes taking values in Euclidean space but this can be replaced by a Hilbert space, a Banach space (both of these are important for SPDEs), a locally compact group, a manifold. Quantised versions are non-commutative Lévy processes on quantum groups.

THEORETICAL

- Lévy processes are simplest generic class of process which have (a.s.) continuous paths interspersed with random jumps of arbitrary size occurring at random times.

- Lévy processes comprise a natural subclass of semimartingales and of Markov processes of Feller type.

- There are many interesting examples - Brownian motion, simple and compound Poisson processes, \( \alpha \)-stable processes, subordinated processes, financial processes, relativistic process, Riemann zeta process . . .
APPLICATIONS

These include:

- Turbulence via Burger’s equation (Bertoin)
- New examples of quantum field theories (Albeverio, Gottshalk, Wu)
- Viscoelasticity (Bouleau)
- Time series - Lévy driven CARMA models (Brockwell, Marquardt)
- Stochastic resonance in non-linear signal processing (Patel, Kosco, Applebaum)
- Finance and insurance (a cast of thousands)

The biggest explosion of activity has been in mathematical finance.

Two major areas of activity are:

- option pricing in incomplete markets.
- interest rate modelling.

Let \((\Omega, \mathcal{F}, P)\) be a probability space, so that \(\Omega\) is a set, \(\mathcal{F}\) is a \(\sigma\)-algebra of subsets of \(\Omega\) and \(P\) is a probability measure defined on \((\Omega, \mathcal{F})\).

Random variables are measurable functions \(X : \Omega \to \mathbb{R}^d\). The law of \(X\) is \(p_X(A) = P(X \in A)\).

\((X_n, n \in \mathbb{N})\) are independent if for all \(i_1, i_2, \ldots, i_r \in \mathbb{N}, A_{i_1}, A_{i_2}, \ldots, A_{i_r} \in \mathcal{B}(\mathbb{R}^d)\),

\[
P(X_{i_1} \in A_{i_1}, X_{i_2} \in A_{i_2}, \ldots, X_{i_r} \in A_r) = P(X_{i_1} \in A_{i_1})P(X_{i_2} \in A_{i_2}) \cdots P(X_{i_r} \in A_r).
\]

If \(X\) and \(Y\) are independent, the law of \(X + Y\) is given by convolution of measures

\[p_{X+Y} = p_X * p_Y, \text{ where } (p_X * p_Y)(A) = \int_{\mathbb{R}^d} p_X(A-y)p_Y(dy).
\]

Notation. Our state space is Euclidean space \(\mathbb{R}^d\). The inner product between two vectors \(x = (x_1, \ldots, x_d)\) and \(y = (y_1, \ldots, y_d)\) is

\[
(x, y) = \sum_{i=1}^{d} x_i y_i.
\]

The associated norm (length of a vector) is

\[|x| = (x, x)^{1/2} = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}.
\]
The *characteristic function* of $X$ is $\phi_X : \mathbb{R}^d \to \mathbb{C}$, where

$$\phi_X(u) = \int_{\mathbb{R}^d} e^{i(u,x)} p_X(dx).$$

**Theorem (Kac’s theorem)**

$X_1, \ldots, X_n$ are independent if and only if

$$\mathbb{E}\left[ \exp \left( i \sum_{j=1}^n (u_j, X_j) \right) \right] = \phi_{X_1}(u_1) \cdots \phi_{X_n}(u_n)$$

for all $u_1, \ldots, u_n \in \mathbb{R}^d$.

Conversely, *Bochner’s theorem* states that if $\phi : \mathbb{R}^d \to \mathbb{C}$ satisfies (1), (2) and is continuous at $u = 0$, then it is the characteristic function of some probability measure $\mu$ on $\mathbb{R}^d$.

$\psi : \mathbb{R}^d \to \mathbb{C}$ is *conditionally positive definite* if for all $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{C}$ for which $\sum_{j=1}^n c_j = 0$ we have

$$\sum_{j,k=1}^n c_j \overline{c_k} \psi(u_j - u_k) \geq 0,$$

for all $u_1, \ldots, u_n \in \mathbb{R}^d$.

Note: *conditionally positive definite* is sometimes called *negative definite*.

$\psi : \mathbb{R}^d \to \mathbb{C}$ will be said to be *hermitian* if $\overline{\psi(u)} = \psi(-u)$, for all $u \in \mathbb{R}^d$.

More generally, the characteristic function of a probability measure $\mu$ on $\mathbb{R}^d$ is

$$\phi_\mu(u) = \int_{\mathbb{R}^d} e^{i(u,x)} \mu(dx).$$

Important properties are:-

- $\phi_\mu(0) = 1$.
- $\phi_\mu$ is *positive definite* i.e. $\sum_{i,j} c_i \overline{c_j} \phi_\mu(u_i - u_j) \geq 0$, for all $c_i \in \mathbb{C}$, $u_i \in \mathbb{R}^d$, $1 \leq i, j \leq n, n \in \mathbb{N}$.
- $\phi_\mu$ is uniformly continuous - Hint: Look at $|\phi_\mu(u+h) - \phi_\mu(u)|$ and use dominated convergence).

Also $\mu \rightarrow \phi_\mu$ is injective.

**Theorem (Schoenberg correspondence)**

$\psi : \mathbb{R}^d \to \mathbb{C}$ is *hermitian and conditionally positive definite* if and only if $e^{i\psi}$ is positive definite for each $t > 0$.

*Proof. We only give the easy part here.*

Suppose that $e^{i\psi}$ is positive definite for all $t > 0$. Fix $n \in \mathbb{N}$ and choose $c_1, \ldots, c_n$ and $u_1, \ldots, u_n$ as above.

We then find that for each $t > 0$,

$$\frac{1}{t} \sum_{j,k=1}^n c_j \overline{c_k} (e^{i\psi(u_j - u_k)} - 1) \geq 0,$$

and so

$$\sum_{j,k=1}^n c_j \overline{c_k} \psi(u_j - u_k) = \lim_{t \to 0} \frac{1}{t} \sum_{j,k=1}^n c_j \overline{c_k} (e^{i\psi(u_j - u_k)} - 1) \geq 0.$$
We say that a Lévy process is infinite divisibility in motion, i.e. infinite divisibility is the underlying probabilistic idea which a Lévy process embodies dynamically.

Let \( \mu \) be a probability measure on \( \mathbb{R}^d \). Define \( \mu^n = \mu \ast \cdots \ast \mu \) \((n \text{ times})\). We say that \( \mu \) has a convolution \( n \text{th root} \), if there exists a probability measure \( \mu^n \) for which \((\mu^n)^n = \mu\).

\( \mu \) is **infinitely divisible** if it has a convolution \( n \text{th root} \) for all \( n \in \mathbb{N} \). In this case \( \mu^n \) is unique.

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**Examples of Infinite Divisibility**

In the following, we will demonstrate infinite divisibility of a random variable \( X \) by finding \( \text{i.i.d.} \ Y_1^{(n)}, \ldots, Y_n^{(n)} \) such that \( X \sim Y_1^{(n)} + \cdots + Y_n^{(n)} \), for each \( n \in \mathbb{N} \).

**Example 1 - Gaussian Random Variables**

Let \( X = (X_1, \ldots, X_d) \) be a random vector.

We say that it is \((\text{non - degenerate})\) \textit{Gaussian} if it there exists a vector \( m \in \mathbb{R}^d \) and a strictly positive-definite symmetric \( d \times d \) matrix \( A \) such that \( X \) has a pdf \((\text{probability density function})\) of the form:

\[
f(x) = \frac{1}{(2\pi)^{d/2} \sqrt{\det(A)}} \exp \left( -\frac{1}{2} (x - m, A^{-1}(x - m)) \right), \tag{1.1}
\]

for all \( x \in \mathbb{R}^d \).

In this case we will write \( X \sim N(m, A) \). The vector \( m \) is the mean of \( X \), so \( m = \mathbb{E}(X) \) and \( A \) is the covariance matrix so that \( A = \mathbb{E}((X - m)(X - m)^T) \).

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**Proof.** If \( \mu \) is infinitely divisible, take \( \phi_n = \phi_{\mu^n} \). Conversely, for each \( n \in \mathbb{N} \), by Fubini’s theorem,

\[
\phi_n (u) = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i(u, y_1 + \cdots + y_n)} \mu_n(dy_1) \cdots \mu_n(dy_n)
= \int_{\mathbb{R}^d} e^{i(u, y)} \mu_n(dy).
\]

But \( \phi_n (u) = \int_{\mathbb{R}^d} e^{i(u, y)} \mu_n(dy) \) and \( \phi \) determines \( \mu \) uniquely. Hence \( \mu = \mu_n^1 \).

\( \square \)
A standard calculation yields

\[ \phi_X(u) = \exp \left\{ i \left( \frac{m}{n} \cdot u - \frac{1}{2} (u, Au) \right) \right\}, \tag{1.2} \]

and hence

\[ (\phi_X(u))^j = \exp \left\{ i \left( \frac{m}{n} \cdot u - \frac{1}{2} (u, Au) \right) \right\}. \]

so we see that \( X \) is infinitely divisible with each \( Y_j^{(n)} \sim N\left( \frac{m}{n}, \frac{1}{n}A \right) \) for each \( 1 \leq j \leq n \).

We say that \( X \) is a standard normal whenever \( X \sim N(0, \sigma^2 I) \) for some \( \sigma > 0 \).

We say that \( X \) is degenerate Gaussian if (1.2) holds with \( \det(A) = 0 \), and these random variables are also infinitely divisible.

\[ \text{Example 3 - Compound Poisson Random Variables} \]

Let \( (Z(n), n \in \mathbb{N}) \) be a sequence of i.i.d. random variables taking values in \( \mathbb{R}^d \) with common law \( \mu_Z \) and let \( N \sim \pi(c) \) be a Poisson random variable which is independent of all the \( Z(n) \)'s. The **compound Poisson random variable** \( X \) is defined as follows:

\[ X := \begin{cases} 0 & \text{if } N = 0 \\ Z(1) + \cdots + Z(N) & \text{if } N > 0. \end{cases} \]

\[ \text{Theorem} \]

For each \( u \in \mathbb{R}^d \),

\[ \phi_X(u) = \exp \left[ \int_{\mathbb{R}^d} (e^{i(u,y)} - 1) c \mu_Z(dy) \right]. \]

\[ \text{Example 2 - Poisson Random Variables} \]

In this case, we take \( d = 1 \) and consider a random variable \( X \) taking values in the set \( n \in \mathbb{N} \cup \{0\} \). We say that is **Poisson** if there exists \( c > 0 \) for which

\[ P(X = n) = \frac{c^n}{n!} e^{-c}. \]

In this case we will write \( X \sim \pi(c) \). We have \( \mathbb{E}(X) = \text{Var}(X) = c \). It is easy to verify that

\[ \phi_X(u) = \exp(c(e^{iu} - 1)), \]

from which we deduce that \( X \) is infinitely divisible with each \( Y_j^{(n)} \sim \pi\left( \frac{c}{n} \right) \), for \( 1 \leq j \leq n, n \in \mathbb{N} \).

\[ \text{Proof.} \] Let \( \phi_Z \) be the common characteristic function of the \( Z_n \)'s. By conditioning and using independence we find,

\[ \phi_X(u) = \sum_{n=0}^{\infty} \mathbb{E}(e^{i(u,Z(1)+\cdots+Z(n))}|N=n)P(N=n) \]

\[ = \sum_{n=0}^{\infty} \mathbb{E}(e^{i(u,Z(1)+\cdots+Z(n))}) e^{-c \frac{c^n}{n!}} \]

\[ = e^{-c} \sum_{n=0}^{\infty} \frac{[c \phi_Z(u)]^n}{n!} \]

\[ = \exp[c(cZ(u) - 1)], \]

and the result follows on writing \( \phi_Z(u) = \int e^{i(u,y)} \mu_Z(dy) \).

\[ \Box \]

If \( X \) is compound Poisson as above, we write \( X \sim \pi(c, \mu_Z) \). It is clearly infinitely divisible with each \( Y_j^{(n)} \sim \pi(\frac{c}{n}, \mu_Z) \), for \( 1 \leq j \leq n \).
The Lévy-Khintchine Formula

de Finetti (1920’s) suggested that the most general infinitely divisible random variable could be written \( X = Y + W \), where \( Y \) and \( W \) are independent, \( Y \sim N(m, A) \), \( W \sim \pi(c, \mu Z) \). Then \( \phi_X(u) = e^{\eta(u)} \), where

\[
\eta(u) = i(m, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d} (e^{i(u, y)} - 1)c\mu_Z(dy).
\]

(1.3)

This is WRONG! \( \nu(\cdot) = c\mu_Z(\cdot) \) is a finite measure here.

Lévy and Khintchine showed that \( \nu \) can be \( \sigma \)-finite, provided it is what is now called a Lévy measure on \( \mathbb{R}^d - \{0\} = \{ x \in \mathbb{R}^d, x \neq 0 \} \), i.e.

\[
\int (|y|^2 \wedge 1)\nu(dy) < \infty,
\]

(1.4)

(where \( a \wedge b := \min\{a, b\} \), for \( a, b \in \mathbb{R} \)). Since \( |y|^2 \wedge \epsilon \leq |y|^2 \wedge 1 \)

whenever \( 0 < \epsilon \leq 1 \), it follows from (1.4) that

\[
\nu((-\epsilon, \epsilon)^c) < \infty \quad \text{for all } \epsilon > 0.
\]

Here is the fundamental result of this lecture:-

**Theorem (Lévy-Khintchine)**

A Borel probability measure \( \mu \) on \( \mathbb{R}^d \) is infinitely divisible if there exists a vector \( b \in \mathbb{R}^d \), a non-negative symmetric \( d \times d \) matrix \( A \) and a Lévy measure \( \nu \) on \( \mathbb{R}^d - \{0\} \) such that for all \( u \in \mathbb{R}^d \),

\[
\phi_{\mu}(u) = \exp \left[ i(b, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} (e^{i(u, y)} - 1 - i(u, y)1\hat{B}^\perp(y))\nu(dy) \right].
\]

(1.5)

where \( \hat{B} = B_1(0) = \{ y \in \mathbb{R}^d; |y| < 1 \} \).

Conversely, any mapping of the form (1.5) is the characteristic function of an infinitely divisible probability measure on \( \mathbb{R}^d \).

The triple \( (b, A, \nu) \) is called the characteristics of the infinitely divisible random variable \( X \). Define \( \eta = \log \phi_{\mu} \), where we take the principal part of the logarithm. \( \eta \) is called the Lévy symbol.

\( \eta \) may also be called the characteristic exponent.

We’re not going to prove this result here. To understand it, it is instructive to let \( (U_n, n \in \mathbb{N}) \) be a sequence of Borel sets in \( B_1(0) \) with \( U_n \downarrow \{0\} \). Observe that

\[
\eta(u) = \lim_{n \to \infty} \eta_n(u) \quad \text{where each}
\]

\[
\eta_n(u) = i \left( b - \int_{U_n^c \cap \hat{B}} y\nu(dy), u \right) - \frac{1}{2}(u, Au) + \int_{U_n^c} (e^{i(u, y)} - 1)\nu(dy),
\]

so \( \eta \) is in some sense (to be made more precise later) the limit of a sequence of sums of Gaussians and independent compound Poissons. Interesting phenomena appear in the limit as we’ll see below.
Stable Laws

This is one of the most important subclasses of infinitely divisible laws. We consider the general central limit problem in dimension \( d = 1 \), so let \((Y_n, n \in \mathbb{N})\) be a sequence of real valued i.i.d. random variables and consider the rescaled partial sums

\[
S_n = \frac{Y_1 + Y_2 + \cdots + Y_n - b_n}{\sigma_n},
\]

where \((b_n, n \in \mathbb{N})\) is an arbitrary sequence of real numbers and \((\sigma_n, n \in \mathbb{N})\) an arbitrary sequence of positive numbers. We are interested in the case where there exists a random variable \(X\) for which

\[
\lim_{n \to \infty} P(S_n \leq x) = P(X \leq x),
\]

for all \(x \in \mathbb{R}\), i.e. \((S_n, n \in \mathbb{N})\) converges in distribution to \(X\). If each \(b_n = nm\) and \(\sigma_n = \sqrt{n}\sigma\) for fixed \(m \in \mathbb{R}, \sigma > 0\) then \(X \sim N(m, \sigma^2)\) by the usual Laplace - de-Moivre central limit theorem.

More generally a random variable is said to be stable if it arises as a limit as in (1.7). It is not difficult to show that (1.7) is equivalent to the following:

There exist real valued sequences \((c_n, n \in \mathbb{N})\) and \((d_n, n \in \mathbb{N})\) with each \(c_n > 0\) such that

\[
X_1 + X_2 + \cdots + X_n = c_nX + d_n
\]

(1.8)

where \(X_1, \ldots, X_n\) are independent copies of \(X\). \(X\) is said to be strictly stable if each \(d_n = 0\).

To see that (1.8) \(\Rightarrow\) (1.7) take each \(Y_j = X_j\), \(b_n = d_n\) and \(\sigma_n = c_n\). In fact it can be shown that the only possible choice of \(c_n\) in (1.8) is \(c_n = \sigma n^{\frac{1}{\alpha}},\) where \(0 < \alpha \leq 2\) and \(\sigma > 0\). The parameter \(\alpha\) plays a key role in the investigation of stable random variables and is called the index of stability.

Note that (1.8) can also be expressed in the equivalent form

\[
\phi_X(u) = e^{iud_n}\phi_X(c_nu),
\]

for each \(u \in \mathbb{R}\).

It follows immediately from (1.8) that all stable random variables are infinitely divisible and the characteristics in the Lévy-Khintchine formula are given as follows:

**Theorem**

If \(X\) is a stable real-valued random variable, then its characteristics must take one of the two following forms.

- **When** \(\alpha = 2\), \(\nu = 0\) (so \(X \sim N(b, A)\)).
- **When** \(\alpha \neq 2\), \(A = 0\) and

\[
\nu(dx) = \frac{c_1}{x^{1+\alpha}}1_{(0,\infty)}(x)dx + \frac{c_2}{|x|^{1+\alpha}}1_{(-\infty,0)}(x)dx,
\]

where \(c_1 \geq 0, c_2 \geq 0\) and \(c_1 + c_2 > 0\).
A careful transformation of the integrals in the Lévy-Khintchine formula gives a different form for the characteristic function which is often more convenient.

**Theorem**

A real-valued random variable $X$ is stable if and only if there exists $\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that for all $u \in \mathbb{R}$,

1. $\phi_X(u) = \exp \left[ i \mu u - \frac{1}{2} \sigma^2 u^2 \right]$

   when $\alpha = 2$

2. $\phi_X(u) = \exp \left[ i \mu u - \sigma^\alpha |u|^\alpha \left( 1 - i \beta \text{sgn}(u) \tan \left( \frac{\pi \alpha}{2} \right) \right) \right]$

   when $\alpha \neq 1, 2$.

It can be shown that $\mathbb{E}(X^2) < \infty$ if and only if $\alpha = 2$ (i.e. $X$ is Gaussian) and $\mathbb{E}(|X|) < \infty$ if and only if $1 < \alpha \leq 2$.

All stable random variables have densities $f_X$, which can in general be expressed in series form. In three important cases, there are closed forms.

- **The Normal Distribution**
  \[
  \alpha = 2, \quad X \sim N(\mu, \sigma^2). \]

- **The Cauchy Distribution**
  \[
  \alpha = 1, \beta = 0 \quad f_X(x) = \frac{\sigma}{\pi [(x - \mu)^2 + \sigma^2]}.
  \]

- **The Lévy Distribution**
  \[
  \alpha = \frac{1}{2}, \beta = 1 \quad f_X(x) = \left( \frac{\sigma}{2\pi} \right)^{1/2} \frac{1}{(x - \mu)^{1/2}} \exp \left( -\frac{\sigma}{2(x - \mu)} \right),
  \]
  for $x > \mu$.

In general the series representations are given in terms of a real valued parameter $\lambda$.

For $x > 0$ and $0 < \alpha < 1$:

\[
  f_X(x, \lambda) = \frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{\Gamma(k\alpha + 1)}{k!}(-x^{-\alpha})^k \sin \left( \frac{k\pi}{2}(\lambda - \alpha) \right)
\]

For $x > 0$ and $1 < \alpha < 2$,

\[
  f_X(x, \lambda) = \frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{\Gamma(k\alpha^{-1} + 1)}{k!}(-x)^k \sin \left( \frac{k\pi}{2\alpha}(\lambda - \alpha) \right)
\]

In each case the formula for negative $x$ is obtained by using

\[
  f_X(-x, \lambda) = f_X(x, -\lambda), \quad \text{for } x > 0.
\]
Note that if a stable random variable is symmetric then Theorem 8 yields

\[ \phi_X(u) = \exp(-\rho \alpha |u|^\alpha) \text{ for all } 0 < \alpha \leq 2, \]

where \( \rho = \sigma(0 < \alpha < 2) \) and \( \rho = \frac{\sigma}{\sqrt{2}} \), when \( \alpha = 2 \), and we will write \( X \sim S_\alpha S \) in this case.

One of the reasons why stable laws are so important in applications is the nice decay properties of the tails. The case \( \alpha = 2 \) is special in that we have exponential decay, indeed for a standard normal \( X \) there is the elementary estimate

\[ P(X > y) \sim e^{-\frac{1}{2}y^2} \text{ as } y \to \infty, \]

When \( \alpha \neq 2 \) we have the slower polynomial decay as expressed in the following,

\[ \lim_{y \to \infty} y^\alpha P(X > y) = C_\alpha \frac{1 + \beta}{2} \sigma^\alpha, \]

\[ \lim_{y \to \infty} y^\alpha P(X < -y) = C_\alpha \frac{1 - \beta}{2} \sigma^\alpha, \]

where \( C_\alpha > 1 \). The relatively slow decay of the tails for non-Gaussian stable laws ("heavy tails") makes them ideally suited for modelling a wide range of interesting phenomena, some of which exhibit "long-range dependence".

We can generalise the definition of stable random variables if we weaken the conditions on the random variables \( (Y(n), n \in \mathbb{N}) \) in the general central limit problem by requiring these to be independent, but no longer necessarily identically distributed. In this case (subject to a technical growth restriction) the limiting random variables are called self-decomposable (or of class L) and they are also infinitely divisible. Alternatively a random variable \( X \) is self-decomposable if and only if for each \( 0 < a < 1 \), there exists a random variable \( Y_a \) which is independent of \( X \) such that

\[ X \overset{d}{=} aX + Y_a \iff \phi_X(u) = \phi_X(au)\phi_{Y_a}(u), \]

for all \( u \in \mathbb{R}^d \).

Deeper mathematical investigations of heavy tails require the mathematical technique of regular variation.

The generalisation of stability to random vectors is straightforward - just replace \( X_1, \ldots, X_n, X \) and each \( d_n \) in (1.8) by vectors and the formula in Theorem 7 extends directly. Note however that when \( \alpha \neq 2 \) in the random vector version of Theorem 7, the Lévy measure takes the form

\[ \nu(dx) = \frac{c}{|x|^{d+\alpha}} dx \]

where \( c > 0 \).
An infinitely divisible law is self-decomposable if and only if the Lévy measure is of the form:

\[ \nu(dx) = \frac{k(x)}{|x|} dx, \]

where \( k \) is decreasing on \((0, \infty)\) and increasing on \((-\infty, 0)\). There has recently been increasing interest in these distributions both from a theoretical and applied perspective. Examples include gamma, Pareto, Student-\( t \), \( F \) and log-normal distributions.

Consider the aggregate behaviour of the sum of a large number of i.i.d. random variables,

- This can be co-operative so that no one r.v. dominates \( \rightarrow \) familiar Gaussian C.L.T.
- This can be dominated by the behaviour of a single random variable \( \rightarrow \) C.L.T. for stable laws.

To capture the second type of behaviour, we say that a non-negative random variable \( X \) is subexponential if as \( x \to \infty \)

\[ P(X_1 + \cdots + X_n > x) \sim P(\max\{X_1, \ldots, X_n\} > x), \]

where \( X_1, \ldots, X_n \) are independent copies of \( X \). A natural and more easily handled subclass of subexponential random variables are those of regular variation of index \( \alpha \).

\( X \in \mathcal{R}_{-\alpha} \) where \( \alpha > 0 \) if

\[ \lim_{x \to \infty} \frac{F_X(cx)}{F_X(x)} = c^\alpha, \]

where \( F_X(x) := P(X > x) \).

e.g. \( X \) Pareto - parameters \( K, \alpha > 0 \),

\[ F_X(x) = \left( \frac{K}{K + x} \right)^\alpha. \]

\( X \) is self-decomposable.

Intuition: “Heavy tails” (subexponentiality) are due to “large jumps”.

Large jumps are governed by the tail of the Lévy measure (see Lecture 3)

Fact (Tail equivalence): If \( X \) is infinitely divisible then \( F_X \in \mathcal{R}_{-\alpha} \) if and only if \( \nu \in \mathcal{R}_{-\alpha} \).

In this case \( \lim_{x \to \infty} \frac{F_X(x)}{\nu(x)} = 1. \)