Definition: Lévy Process

Let $X = (X(t), t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$.

We say that it has **independent increments** if for each $n \in \mathbb{N}$ and each $0 \leq t_1 < t_2 < \cdots < t_{n+1} < \infty$, the random variables

$$X(t_{j+1}) - X(t_j), 1 \leq j \leq n$$

are independent

and it has **stationary increments** if each

$$X(t_{j+1}) - X(t_j) \overset{d}{=} X(t_{j+1} - t_j) - X(0).$$

We say that $X$ is a **Lévy process** if

- (L1) Each $X(0) = 0$ (a.s.),
- (L2) $X$ has independent and stationary increments,
- (L3) $X$ is **stochastically continuous** i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Note that in the presence of (L1) and (L2), (L3) is equivalent to the condition

$$\lim_{t \downarrow 0} P(|X(t)| > a) = 0.$$
Proof. For each $n \in \mathbb{N}$, we can write
\[ X(t) = Y_1^{(n)}(t) + \cdots + Y_n^{(n)}(t) \]
where each $Y_k^{(n)}(t) = X\left(\frac{kt}{n}\right) - X\left(\frac{(k-1)t}{n}\right)$.
The $Y_k^{(n)}(t)$'s are i.i.d. by (L2).
From Lecture 1 we can write $\phi_X(t)(u) = e^{i\eta(t,u)}$ for each $t \geq 0$, $u \in \mathbb{R}^d$, where each $\eta(t,\cdot)$ is a Lévy symbol.

We will define the Lévy symbol and the characteristics of a Lévy process $X$ to be those of the random variable $X(1)$. We will sometimes write the former as $\eta_X$ when we want to emphasise that it belongs to the process $X$.

We now have the Lévy-Khinchine formula for a Lévy process $X = (X(t), t \geq 0)$:
\[
\mathbb{E}(e^{i(u,X(t))}) = \exp\left\{ t \left[ i(b,u) - \frac{1}{2}(u,Au) \right] + \int_{\mathbb{R}^d-\{0\}} \left( e^{i(u,y)} - 1 - i(u,y)1_B(y) \right) \nu(dy) \right\}, (2.1)
\]
for each $t \geq 0$, $u \in \mathbb{R}^d$, where $(b, A, \nu)$ are the characteristics of $X(1)$. We will define the Lévy symbol and the characteristics of a Lévy process $X$ to be those of the random variable $X(1)$. We will sometimes write the former as $\eta_X$ when we want to emphasise that it belongs to the process $X$. 

\[ \phi_{X(t)}(u) = e^{i\eta(u)}; \]
for each $u \in \mathbb{R}^d$, $t \geq 0$, where $\eta$ is the Lévy symbol of $X(1)$. 

Proof. Suppose that $X$ is a Lévy process and for each $u \in \mathbb{R}^d$, $t \geq 0$, define $\phi_u(t) = \phi_{X(t)}(u)$
then by (L2) we have for all $s \geq 0$,
\[
\phi_u(t+s) = \mathbb{E}(e^{i(u,X(t+s))}) \\
= \mathbb{E}(e^{i(u,X(t+s)-X(s))}e^{i(u,X(s))}) \\
= \mathbb{E}(e^{i(u,X(t+s)-X(s))})\mathbb{E}(e^{i(u,X(s))}) \\
= \phi_u(t)\phi_u(s) \ldots (i)
\]
Now $\phi_u(0) = 1 \ldots (ii)$ by (L1), and the map $t \to \phi_u(t)$ is continuous.
However the unique continuous solution to (i) and (ii) is given by
$\phi_u(t) = e^{i\alpha(u)}$, where $\alpha : \mathbb{R}^d \to \mathbb{C}$. Now by Theorem 1, $X(1)$ is infinitely divisible, hence $\alpha$ is a Lévy symbol and the result follows. 
\[ \square \]
Let \( p_t \) be the law of \( X(t) \), for each \( t \geq 0 \). By (L2), we have for all \( s, t \geq 0 \) that:

\[
p_{t+s} = p_t * p_s.
\]

By (L3), we have \( p_t \xrightarrow{w} \delta_0 \) as \( t \to 0 \), i.e. \( \lim_{t \to 0} f(x)p_t(dx) = f(0) \).

\((p_t, t \geq 0)\) is a weakly continuous convolution semigroup of probability measures on \( \mathbb{R}^d \).

Conversely, given any such semigroup, we can always construct a Lévy process on path space via Kolmogorov’s construction.

Informally, we have the following asymptotic relationship between the law of a Lévy process and its Lévy measure:

\[
\nu = \lim_{t \downarrow 0} \frac{p_t}{t}.
\]

More precisely

\[
\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} f(x)p_t(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx), \tag{2.2}
\]

for bounded, continuous functions \( f \) which vanish in some neighborhood of the origin.

### Examples of Lévy Processes

**Example 1, Brownian Motion and Gaussian Processes**

A (standard) Brownian motion in \( \mathbb{R}^d \) is a Lévy process \( B = (B(t), t \geq 0) \) for which

1. \( B(t) \sim N(0, tI) \) for each \( t \geq 0 \),
2. \( B \) has continuous sample paths.

It follows immediately from (B1) that if \( B \) is a standard Brownian motion, then its characteristic function is given by

\[
\phi_B(t)(u) = \exp\left\{-\frac{1}{2}t|u|^2\right\},
\]

for each \( u \in \mathbb{R}^d, t \geq 0 \).

We introduce the marginal processes \( B_i = (B_i(t), t \geq 0) \) where each \( B_i(t) \) is the \( i \)th component of \( B(t) \), then it is not difficult to verify that the \( B_i \)'s are mutually independent Brownian motions in \( \mathbb{R} \). We will call these one-dimensional Brownian motions in the sequel.

Brownian motion has been the most intensively studied Lévy process. In the early years of the twentieth century, it was introduced as a model for the physical phenomenon of Brownian motion by Einstein and Smoluchowski and as a description of the dynamical evolution of stock prices by Bachelier.
The theory was placed on a rigorous mathematical basis by Norbert Wiener in the 1920’s. We could try to use the Kolmogorov existence theorem to construct one-dimensional Brownian motion from the following prescription on cylinder sets of the form $I_{t_1}^{t_n} = \{ \omega \in \Omega; \omega(t_1) \in [a_1, b_1], \ldots, \omega(t_n) \in [a_n, b_n] \}$ where $H = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and we have taken $\Omega$ to be the set of all mappings from $\mathbb{R}^+$ to $\mathbb{R}$:

$$P(I_{t_1}^{t_n})$$

$$= \int_H \frac{1}{\sqrt{(2\pi)^{n}}} \exp\left( -\frac{1}{2} \left( \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \cdots \right. \right.$$

$$+ \left. \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}} \right) dx_1 \cdots dx_n.$$

However there there is then no guarantee that the paths are continuous.

We list a number of useful properties of Brownian motion in the case $d = 1$.

- Brownian motion is locally Hölder continuous with exponent $\alpha$ for every $0 < \alpha < \frac{1}{2}$ i.e. for every $T > 0, \omega \in \Omega$ there exists $K = K(T, \omega)$ such that

$$|B(t)(\omega) - B(s)(\omega)| \leq K|t - s|^{\alpha},$$

for all $0 \leq s < t \leq T$.

- The sample paths $t \to B(t)(\omega)$ are almost surely nowhere differentiable.

- For any sequence, $(t_n, n \in \mathbb{N})$ in $\mathbb{R}^+$ with $t_n \uparrow \infty$,

$$\lim_{n \to \infty} B(t_n) = -\infty \quad \text{a.s.} \quad \limsup_{n \to \infty} B(t_n) = \infty \quad \text{a.s.}$$

- The law of the iterated logarithm:

$$P\left( \limsup_{t \uparrow 0} \frac{B(t)}{\sqrt{2t \log \log (\frac{1}{t})}} \right) = 1.$$

The literature contains a number of ingenious methods for constructing Brownian motion. One of the most delightful of these (originally due to Paley and Wiener) obtains this, in the case $d = 1$, as a random Fourier series for $0 \leq t \leq 1$:

$$B(t) = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \sin\left( \pi t (n + \frac{1}{2}) \right) \frac{\xi(n)}{n + \frac{1}{2}},$$

for each $t \geq 0$, where $(\xi(n), n \in \mathbb{N})$ is a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables.
Let $A$ be a non-negative symmetric $d \times d$ matrix and let $\sigma$ be a square root of $A$ so that $\sigma$ is a $d \times m$ matrix for which $\sigma \sigma^T = A$. Now let $b \in \mathbb{R}^d$ and let $B$ be a Brownian motion in $\mathbb{R}^m$. We construct a process $C = (C(t), t \geq 0)$ in $\mathbb{R}^d$ by

$$C(t) = bt + \sigma B(t),$$

(2.3)

then $C$ is a Lévy process with each $C(t) \sim N(tb, tA)$. It is not difficult to verify that $C$ is also a Gaussian process, i.e. all its finite dimensional distributions are Gaussian. It is sometimes called Brownian motion with drift. The Lévy symbol of $C$ is

$$\eta_C(u) = i(b, u) - \frac{1}{2}(u, Au).$$

In fact a Lévy process has continuous sample paths if and only if it is of the form (2.3).

Example 2 - The Poisson Process

The Poisson process of intensity $\lambda > 0$ is a Lévy process $N$ taking values in $\mathbb{N} \cup \{0\}$ wherein each $N(t) \sim \pi(\lambda t)$ so we have

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},$$

for each $n = 0, 1, 2, \ldots$.

The Poisson process is widely used in applications and there is a wealth of literature concerning it and its generalisations.

We define non-negative random variables $(T_n, \mathbb{N} \cup \{0\})$ (usually called waiting times) by $T_0 = 0$ and for $n \in \mathbb{N}$,

$$T_n = \inf\{t \geq 0; N(t) = n\},$$

then it is well known that the $T_n$’s are gamma distributed. Moreover, the inter-arrival times $T_n - T_{n-1}$ for $n \in \mathbb{N}$ are i.i.d. and each has exponential distribution with mean $\frac{1}{\lambda}$. The sample paths of $N$ are clearly piecewise constant with “jump” discontinuities of size 1 at each of the random times $(T_n, n \in \mathbb{N})$.
For later work it is useful to introduce the compensated Poisson process \( \tilde{N} = (\tilde{N}(t), t \geq 0) \) where each \( \tilde{N}(t) = N(t) - \lambda t \). Note that \( \mathbb{E}(\tilde{N}(t)) = 0 \) and \( \mathbb{E}(\tilde{N}(t)^2) = \lambda t \) for each \( t \geq 0 \).

Example 3 - The Compound Poisson Process

Let \((Z(n), n \in \mathbb{N})\) be a sequence of i.i.d. random variables taking values in \( \mathbb{R}^d \) with common law \( \mu_Z \) and let \( N \) be a Poisson process of intensity \( \lambda \) which is independent of all the \( Z(n) \)'s. The compound Poisson process \( Y \) is defined as follows:

\[
Y(t) := \begin{cases} 
0 & \text{if } N(t) = 0 \\
Z(1) + \cdots + Z(N(t)) & \text{if } N(t) > 0,
\end{cases}
\]

for each \( t \geq 0 \), so each \( Y(t) \sim \pi(\lambda t, \mu_Z) \).

From the work of Lecture 1, \( Y \) has Lévy symbol

\[
\eta_Y(u) = \left[ \int (e^{i(u,y)} - 1) \lambda \mu_Z(dy) \right].
\]

Again the sample paths of \( Y \) are piecewise constant with “jump discontinuities” at the random times \( (T(n), n \in \mathbb{N}) \), however this time the size of the jumps is itself random, and the jump at \( T(n) \) can be any value in the range of the random variable \( Z(n) \).

Simulation of a compound Poisson process with \( N(0, 1) \) summands (\( \lambda = 1 \)).
Example 4 - Interlacing Processes

Let $C$ be a Gaussian Lévy process as in Example 1 and $Y$ be a compound Poisson process as in Example 3, which is independent of $C$.

Define a new process $X$ by

$$X(t) = C(t) + Y(t),$$

for all $t \geq 0$, then it is not difficult to verify that $X$ is a Lévy process with Lévy symbol

$$\eta_X(u) = i(b, u) - \frac{1}{2} (u, Au) + \left[ \int (e^{i(u,y)} - 1) \lambda \mu_Z(dy) \right].$$

Using the notation of Examples 2 and 3, we see that the paths of $X$ have jumps of random size occurring at random times.

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Example 5 - Stable Lévy Processes

A stable Lévy process is a Lévy process $X$ in which the Lévy symbol is that of a given stable law. So, in particular, each $X(t)$ is a stable random variable. For example, we have the rotationally invariant case whose Lévy symbol is given by

$$\eta(u) = -\sigma^\alpha |u|^{\alpha},$$

where $\alpha$ is the index of stability ($0 < \alpha \leq 2$). One of the reasons why these are important in applications is that they display self-similarity.

In general, a stochastic process $Y = (Y(t), t \geq 0)$ is self-similar with Hurst index $H > 0$ if the two processes $(Y(at), t \geq 0)$ and $(a^H Y(t), t \geq 0)$ have the same finite-dimensional distributions for all $a \geq 0$. By examining characteristic functions, it is easily verified that a rotationally invariant stable Lévy process is self-similar with Hurst index $H = \frac{1}{\alpha}$, so that e.g. Brownian motion is self-similar with $H = \frac{1}{2}$. A Lévy process $X$ is self-similar if and only if each $X(t)$ is strictly stable.
Question: When does a Lévy process have a density $f_t$ for all $t > 0$ so that for all Borel sets $B$:

$$P(X_t \in B) = p_t(B) = \int_B f_t(x) \, dx$$

In general, a random variable has a continuous density if its characteristic function is integrable and in this case, the density is the Fourier transform of the characteristic function.

So for Lévy processes, if for all $t > 0$,

$$\int_{\mathbb{R}^d} |e^{it\eta(u)}| \, du = \int_{\mathbb{R}^d} e^{i\Re(\eta(u))} \, du < \infty$$

we then have

$$f_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{it\eta(u) - i(x, u)} \, du.$$

Every Lévy process with a non-degenerate Gaussian component has a density.

In this case

$$\Re(\eta(u)) = -\frac{1}{2}(u, Au) + \int_{\mathbb{R}^d \setminus \{0\}} (\cos(u, y) - 1) \nu(dy),$$

and so

$$\int_{\mathbb{R}^d} e^{it\Re(\eta(u))} \, du \leq \int_{\mathbb{R}^d} e^{-\frac{1}{2}(u, Au)} \, du < \infty,$$

using $(u, Au) \geq \lambda |u|^2$ where $\lambda > 0$ is smallest eigenvalue of $A$.

For examples where densities exist for $A = 0$ with $d = 1$: if $X$ is $\alpha$-stable, it has a density since for all $1 \leq \alpha \leq 2$:

$$\int_{|u| \geq 1} e^{-|u|^{\alpha}} \, du \leq \int_{|u| \geq 1} e^{-|u|} \, du < \infty,$$

and for $0 \leq \alpha < 1$:

$$\int_{\mathbb{R}} e^{-|u|^{\alpha}} \, du = \frac{2}{\alpha} \int_0^{\infty} e^{-y^{\frac{1}{\alpha}-1}} \, dy < \infty.$$
In general, a sufficient condition for a density is

- $\nu(\mathbb{R}^d) = \infty$
- $\tilde{\nu}^m$ is absolutely continuous with respect to Lebesgue measure for some $m \in \mathbb{N}$ where

$$\tilde{\nu}(A) = \int_A (|x|^2 \wedge 1) \nu(dx).$$

A Lévy process has a Lévy density $g_\nu$ if its Lévy measure $\nu$ is absolutely continuous with respect to Lebesgue measure, then $g_\nu$ is defined to be the Radon-Nikodym derivative $\frac{d\nu}{dx}$.

A process may have a Lévy density but not have a density.

**Example.** Let $X$ be a compound Poisson process with each $X(t) = Y_1 + Y_2 + \cdots + Y_{N(t)}$ wherein each $Y_j$ has a density $f_Y$, then

$$g_\nu = \lambda f_Y$$

is the Lévy density.

But

$$P(Y(t) = 0) \geq P(N(t) = 0) = e^{-\lambda t} > 0,$$

so $p_t$ has an atom at $\{0\}$.

We have $p_t(A) = e^{-\lambda t} \delta_0(A) + \int_A f^{\text{ac}}(x) dx$, where for $x \neq 0$

$$f^{\text{ac}}(x) = e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} f_Y^n(x).$$

$f^{\text{ac}}(x)$ is the conditional density of $X(t)$ given that it jumps at least once between 0 and $t$.

In this case, (2.2) takes the precise form (for $x \neq 0$)

$$g_\nu(x) = \lim_{t \downarrow 0} \frac{f^{\text{ac}}(x)}{t}.$$

Subordinators

A subordinator is a one-dimensional Lévy process which is increasing a.s. Such processes can be thought of as a random model of time evolution, since if $T = (T(t), t \geq 0)$ is a subordinator we have

$$T(t) \geq 0 \text{ for each } t > 0 \text{ a.s. and } T(t_1) \leq T(t_2) \text{ whenever } t_1 \leq t_2 \text{ a.s.}$$

Now since for $X(t) \sim N(0, At)$ we have

$$P(X(t) \geq 0) = P(X(t) \leq 0) = \frac{1}{2},$$

it is clear that such a process cannot be a subordinator.
Theorem

If $T$ is a subordinator then its Lévy symbol takes the form

$$\eta(u) = iBu + \int_{(0,\infty)} (e^{iuy} - 1)\lambda(dy), \quad (2.4)$$

where $b \geq 0$, and the Lévy measure $\lambda$ satisfies the additional requirements

$$\lambda(-\infty, 0) = 0 \quad \text{and} \quad \int_{(0,\infty)} (y \wedge 1) \lambda(dy) < \infty.$$

Conversely, any mapping from $\mathbb{R}^d \to \mathbb{C}$ of the form (2.4) is the Lévy symbol of a subordinator.

We call the pair $(b, \lambda)$, the characteristics of the subordinator $T$.

For each $t \geq 0$, the map $u \to \mathbb{E}(e^{iuT(t)})$ can be analytically continued to the region $\{iu, u > 0\}$ and we then obtain the following expression for the Laplace transform of the distribution

$$\mathbb{E}(e^{-uT(t)}) = e^{-t\psi(u)},$$

where $\psi(u) = -\eta(iu) = Bu + \int_{(0,\infty)} (1 - e^{-uy})\lambda(dy) \quad (2.5)$

for each $t, u \geq 0$.

This is much more useful for both theoretical and practical application than the characteristic function.

The function $\psi$ is usually called the Laplace exponent of the subordinator.

Examples of Subordinators

(1) The Poisson Case

Poisson processes are clearly subordinators. More generally a compound Poisson process will be a subordinator if and only if the $Z(n)$'s are all $\mathbb{R}^+$ valued.

(2) $\alpha$-Stable Subordinators

Using straightforward calculus, we find that for $0 < \alpha < 1, u \geq 0$,

$$u^\alpha = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-ux}) \frac{dx}{x^{1+\alpha}}.$$

Hence for each $0 < \alpha < 1$ there exists an $\alpha$-stable subordinator $T$ with Laplace exponent

$$\psi(u) = u^\alpha.$$

and the characteristics of $T$ are $(0, \lambda)$ where $\lambda(dx) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{dx}{x^{1+\alpha}}$.

Note that when we analytically continue this to obtain the Lévy symbol we obtain the form given in Lecture 1 for stable laws with $\mu = 0, \beta = 1$ and $\sigma^\alpha = \cos\left(\frac{\alpha\pi}{2}\right)$.
(3) **The Lévy Subordinator**

The $\frac{1}{
2\text{-st}}$-stable subordinator has a density given by the Lévy distribution (with $\mu = 0$ and $\sigma = \frac{\pi^2}{2}$)

$$f_T(t)(s) = \left(\frac{t}{2\sqrt{s}}\right) s^{-\frac{3}{2}} e^{-\frac{\pi^2 s}{4t}},$$

for $s > 0$. The Lévy subordinator has a nice probabilistic interpretation as a first hitting time for one-dimensional standard Brownian motion $(B(t), t \geq 0)$,

$$T(t) = \inf\left\{ s > 0; B(s) = \frac{t}{\sqrt{2}} \right\}.$$  \hspace{1cm} (2.6)

To show directly that for each $t \geq 0$,

$$\mathbb{E}(e^{-uT(t)}) = \int_0^{\infty} e^{-us} f_T(t)(s) \, ds = e^{-\frac{u^2}{2}},$$

write $g_t(u) = \mathbb{E}(e^{-uT(t)})$. Differentiate with respect to $u$ and make the substitution $x = \frac{u^2}{4t}$ to obtain the differential equation $g_t'(u) = -\frac{t}{2\sqrt{s}} g_t(u)$. Via the substitution $y = \frac{t}{2\sqrt{s}}$ we see that $g_t(0) = 1$ and the result follows.

(4) **Inverse Gaussian Subordinators**

We generalise the Lévy subordinator by replacing Brownian motion by the Gaussian process $C = (C(t), t \geq 0)$ where each $C(t) = B(t) + \mu t$ and $\mu \in \mathbb{R}$. The **inverse Gaussian subordinator** is defined by

$$T(t) = \inf\{ s > 0; C(s) = \delta t \}$$

where $\delta > 0$ and is so-called since $t \rightarrow T(t)$ is the generalised inverse of a Gaussian process.

Using martingale methods, we can show that for each $t, u > 0$,

$$\mathbb{E}(e^{-uT(t)}) = e^{-t\delta(\sqrt{2u+\mu^2})},$$  \hspace{1cm} (2.7)

In fact each $T(t)$ has a density:

$$f_T(t)(s) = \frac{\delta t}{\sqrt{2\pi}} e^{-\delta t} s^{-\frac{3}{2}} \exp\left\{ -\frac{1}{2}(t^2 \delta^2 s^{-1} + \mu^2 s) \right\},$$  \hspace{1cm} (2.8)

for each $s, t \geq 0$.

In general any random variable with density $f_T(t)$ is called an **inverse Gaussian** and denoted as $\text{IG}(\delta, \mu)$.

(5) **Gamma Subordinators**

Let $(T(t), t \geq 0)$ be a **gamma process** with parameters $a, b > 0$ so that each $T(t)$ has density

$$f_T(t)(x) = \frac{b^a t}{\Gamma(at)} x^{a-1} e^{-bx},$$

for $x \geq 0$; then it is easy to verify that for each $u \geq 0$,

$$\int_0^{\infty} e^{-ux} f_T(t)(x) \, dx = \left(1 + \frac{u}{b}\right)^{-at} = \exp\left(-t a \log\left(1 + \frac{u}{b}\right)\right).$$

From here it is a straightforward exercise in calculus to show that

$$\int_0^{\infty} e^{-ux} f_T(t)(x) \, dx = \exp\left[-t \int_0^{\infty} (1 - e^{-ux}) ax^{-1} e^{-bx} \, dx\right].$$
From this we see that \((T(t), t \geq 0)\) is a subordinator with \(b = 0\) and 
\[ \lambda(dx) = ax^{-1}e^{-bx}dx. \] Moreover \(\psi(u) = a \log(1 + \frac{u}{b})\) is the associated Bernstein function (see below).

Before we go further into the probabilistic properties of subordinators we’ll make a quick diversion into analysis.

Let \(f \in C^\infty([0, \infty))\). We say it is completely monotone if \((-1)^n f^{(n)} \geq 0\) for all \(n \in \mathbb{N}\), and a Bernstein function if \(f \geq 0\) and \((-1)^n f^{(n)} \leq 0\) for all \(n \in \mathbb{N}\).

Theorem

1. \(f\) is a Bernstein function if and only if the mapping \(x \to e^{-tf(x)}\) is completely monotone for all \(t \geq 0\).
2. \(f\) is a Bernstein function if and only if it has the representation 
\[ f(x) = a + bx + \int_0^\infty (1 - e^{-yx})\lambda(dy), \] 
for all \(x > 0\) where \(a, b \geq 0\) and \(\int_0^\infty (y \wedge 1)\lambda(dy) < \infty\).
3. \(g\) is completely monotone if and only if there exists a measure \(\mu\) on \([0, \infty)\) for which 
\[ g(x) = \int_0^\infty e^{-xy}\mu(dy). \]
To interpret this theorem, first consider the case \( a = 0 \). In this case, if we compare the statement of Theorem 4 with equation (2.5), we see that there is a one to one correspondence between Bernstein functions for which \( \lim_{x \to 0} f(x) = 0 \) and Laplace exponents of subordinators. The Laplace transforms of the laws of subordinators are always completely monotone functions and a subclass of all possible measures \( \mu \) appearing in Theorem 4 (3) is given by all possible laws \( \rho_{T(t)} \) associated to subordinators. A general Bernstein function with \( a > 0 \) can be given a probabilistic interpretation by means of “killing”.

One of the most important probabilistic applications of subordinators is to “time change”. Let \( X \) be an arbitrary Lévy process and let \( T \) be a subordinator defined on the same probability space as \( X \) such that \( X \) and \( T \) are independent. We define a new stochastic process \( Z = (Z(t), t \geq 0) \) by the prescription

\[
Z(t) = X(T(t)),
\]

for each \( t \geq 0 \) so that for each \( \omega \in \Omega \), \( Z(t)(\omega) = X(T(t)(\omega))(\omega) \). The key result is then the following.

**Theorem**

\( Z \) is a Lévy process.

We compute the Lévy symbol of the subordinated process \( Z \).

**Theorem**

\[
\eta_Z = -\psi_{T \circ (-\eta_X)}.
\]

**Proof.** For each \( u \in \mathbb{R}^d, t \geq 0 \),

\[
e^{i\eta_Z(u)} = \mathbb{E}(e^{i(u,Z(t))}) = \mathbb{E}(e^{i(u,X(T(t)))}) = \int \mathbb{E}(e^{i(u,X(s))})\rho_{T(t)}(ds) = \int e^{i\eta_X(u)}\rho_{T(t)}(ds) = \mathbb{E}(e^{i(-\eta_X(u))T(t)}) = e^{-t\psi_{T(-\eta_X(u))}}.
\]

Example: From Brownian Motion to \( 2\alpha \)-stable Processes

Let \( T \) be an \( \alpha \)-stable subordinator (with \( 0 < \alpha < 1 \)) and \( X \) be a \( d \)-dimensional Brownian motion with covariance \( A = 2I \), which is independent of \( T \). Then for each \( s \geq 0, u \in \mathbb{R}^d, \psi_{T(s)} = s^\alpha \) and \( \eta_X(u) = -|u|^2 \), and hence \( \eta_Z(u) = -|u|^{2\alpha} \), i.e. \( Z \) is a rotationally invariant \( 2\alpha \)-stable process.

In particular, if \( d = 1 \) and \( T \) is the Lévy subordinator, then \( Z \) is the Cauchy process, so each \( Z(t) \) has a symmetric Cauchy distribution with parameters \( \mu = 0 \) and \( \sigma = 1 \). It is interesting to observe from (2.6) that \( Z \) is constructed from two independent standard Brownian motions.
Examples of subordinated processes have recently found useful applications in mathematical finance. We briefly mention two interesting cases:

(i) The Variance Gamma Process
In this case \( Z(t) = B(T(t)) \), for each \( t \geq 0 \), where \( B \) is a standard Brownian motion and \( T \) is an independent gamma subordinator. The name derives from the fact that, in a formal sense, each \( Z(t) \) arises by replacing the variance of a normal random variable by a gamma random variable. Using Theorem 6, a simple calculation yields

\[
\Phi_{Z(t)}(u) = \left(1 + \frac{u^2}{2b}\right)^{-at},
\]

for each \( t \geq 0, u \in \mathbb{R} \), where \( a \) and \( b \) are the usual parameters which determine the gamma process. It is an easy exercise in manipulating characteristic functions to compute the alternative representation:

\[
Z(t) = G(t) - L(t),
\]

where \( G \) and \( L \) are independent gamma subordinators each with parameters \( \sqrt{2b} \) and \( a \). This yields a nice financial representation of \( Z \) as a difference of independent “gains” and “losses”. From this representation, we can compute that \( Z \) has a Lévy density

\[
g_v(x) = \frac{a}{|x|} (e^{\sqrt{2b}x} 1_{(-\infty,0)}(x) + e^{-\sqrt{2b}x} 1_{(0,\infty)}(x)).
\]

(ii) The Normal Inverse Gaussian Process
In this case \( Z(t) = C(T(t)) + \mu t \) for each \( t \geq 0 \) where each \( C(t) = B(t) + \beta t, \) with \( \beta \in \mathbb{R} \). Here \( T \) is an inverse Gaussian subordinator, which is independent of \( B \), and in which we write the parameter \( \gamma = \sqrt{\alpha^2 - \beta^2} \), where \( \alpha \in \mathbb{R} \) with \( \alpha^2 \geq \beta^2 \). \( Z \) depends on four parameters and has characteristic function

\[
\Phi_{Z(t)}(\alpha, \beta, \delta, \mu)(u) = \exp \{ \delta t (\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}) + i\mu tu \}
\]

for each \( u \in \mathbb{R}, t \geq 0 \). Here \( \delta > 0 \) is as in (2.7).
Each \( Z(t) \) has a density given by

\[
f_{Z(t)}(x) = C(\alpha, \beta, \delta, \mu; t) q \left( \frac{x - \mu t}{\delta t} \right)^{-1} K_1 \left( \delta t \alpha q \left( \frac{x - \mu t}{\delta t} \right) \right) e^{\beta x},
\]

for each \( x \in \mathbb{R} \), where

\[
q(x) = \sqrt{1 + x^2}, \ C(\alpha, \beta, \delta, \mu; t) = \pi^{-1} \alpha e^{\beta t \sqrt{\alpha^2 - \beta^2}} e^{-\beta \mu t} \text{ and } K_1 \text{ is a Bessel function of the third kind.}