Chapter 2

Sequences

2.1 Convergence of Sequences

A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$. We write $f(1) = a_1, f(2) = a_2$, and in general $f(n) = a_n$. We usually identify the sequence with the range of f, which is written as $(a_1, a_2, a_3, \dots)$ or $(a_n, n \in \mathbb{N})$, or just simply (a_n). Make sure you distinguish the sequence (a_n) from its nth term a_n.

Sequences are important for a number of reasons. They are the simplest context in which we can study the notion of convergence, and they will play a vital role later when we study limits of functions, continuity, and infinite series. You've already met the definition of convergence of a sequence in MAS114, and investigated some of its consequences. We'll revise these ideas, and then go further.

Consider the sequence $(1/n) = (1, 1/2, 1/3, 1/4, 1/5, \ldots)$. It is clear that as n gets larger and larger, the terms get closer and closer to zero, but no value of n will give $1/n = 0$. Of course, zero is the limit, but how do we make this precise? More generally, suppose that we have a sequence (a_n), and we have good reason to believe that its limit is the real number l. How do we prove that l really is the limit? We need to understand more deeply what a limit really is. Although a_n can (typically) never reach l, we can make the distance between them arbitrarily small. Recall from Problem 8 that if $\epsilon > 0$ (which we may think of as being a very small number) then

$$|a_n - l| < \epsilon \text{ if and only if } l - \epsilon < a_n < l + \epsilon.$$

We want to say that as n gets very large, $|a_n - l|$ gets very small. How small? As small as we like! To do this we need to go quite far along the sequence. So we fix $N \in \mathbb{N}$ that may be very large. Then we require that beyond N, i.e. for all $n > N$, $|a_n - l| < \epsilon$.

How small do we take ϵ? How large do we take N? Suppose we have two friends: Kim and Kanye. Kim doesn't think that (a_n) converges to l. She says what if I take $\epsilon = 0.001$, or 0.00001 or 3.18×10^{-700}? Kanye says, no matter how small an ϵ you give me, I can always find an N so that for all $n > N, |a_n - l| < \epsilon$. But he has to prove this. Only then will Kim be satisfied. This leads to the following:

Definition 2.1 A sequence (a_n) is said to *converge* to a limit $l \in \mathbb{R}$ if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that for all $n > N$, we have $|a_n - l| < \epsilon$.

Notation. If (a_n) converges to l, we write

- $\lim_{n\to\infty} a_n = l$,
- or $a_n \to l$ as $n \to \infty$,
- or $a_n \overset{n\to\infty}{\longrightarrow} l$.

Example 2.1 To show $\lim_{n\to\infty} 1/n = 0$.

Before we give a rigorous proof, lets just do a quick experiment. Suppose we take $\epsilon = 10^{-3}$. We'll find an N so that $n > N \Rightarrow 1/n = |1/n - 0| < \epsilon$ for all $n > N$. In fact, its not hard to see that $N = 1000$ will do the trick, but we should always be aware that although such numerical experiments might help us gain some insight, they can never give a proof (no matter how small you take a particular ϵ). This is because we need to be able to implement the process for *all* $\epsilon > 0$. Note that the N you need depends on the choice of ϵ; the smaller your ϵ, the larger your N.

To *prove* that $\lim_{n\to\infty} 1/n = 0$, we use the Archimedean property of the real numbers (see the remarks after Theorem 1.4.4) to deduce that given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Then for all $n > N, 1/n < 1/N < \epsilon$, and we are done.

Note that we needed the completeness property of $\mathbb{R}$ to prove Theorem 1.4.4 which we have just used. Also, although we've shown that $(1/n)$ converges to zero, how do we know that it is the only limit?

Theorem 2.1.1. *If a sequence converges to a limit, then that limit is unique. More precisely if (a_n) is a sequence such that $\lim_{n\to\infty} a_n = l$, and also $\lim_{n\to\infty} a_n = l'$ then $l = l'$.*

Proof. We'll seek a proof by contradiction. Suppose $l \neq l'$. By definition of convergence, given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|a_n - l| < \frac{\epsilon}{2}$, and there also exists $M \in \mathbb{N}$ such that if $n > M$ then

$|a_n - l'| < \frac{\epsilon}{2}$. Let $n > \max\{M, N\}$. By the triangle inequality,

$$\begin{aligned} |l - l'| &= |l - a_n + a_n - l'| \\ &\leq |l - a_n| + |a_n - l'| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So we've shown that for any $\epsilon > 0$, $|l - l'| < \epsilon$ so that $|l - l'|$ is smaller than any positive number. By definition of the modulus, $|l - l'| \geq 0$ and so the only possibility is that $|l - l'| = 0$, i.e. $l = l'$ and this is our required contradiction. $\square$

Example 2.2 To prove that $\lim_{n\to\infty} r^n = 0$, whenever $0 \leq r < 1$.

The result is obvious when $r = 0$, so assume $r > 0$. Since $r < 1, 1/r > 1$ so we can write $1/r = 1 + h$, where $h > 0$. Now use Bernoulli's inequality (see Problem 5) to write

$$1/r^n = (1 + h)^n \geq 1 + nh.$$

Hence $r^n \leq (1 + nh)^{-1}$. To obtain the required result we need to show that given $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $n > N \Rightarrow r^n < \epsilon$. Now we need to do some algebra. To find N such that $(1 + Nh)^{-1} < \epsilon$ will do the trick. Check that this requires that $N > \dfrac{1-\epsilon}{h\epsilon}$. The Archimedean property tells us that such an N exists. Then for all $n > N$,

$$r^n \leq \frac{1}{1 + nh} < \frac{1}{1 + Nh} < \epsilon,$$

and we are done.

Definition A sequence which fails to converge is said to *diverge*.

Divergent sequences may display different types of behaviour. For example, a sequence (a_n) is said to *diverge to* ∞ (respectively, *diverge to* $-\infty$), if given any $K > 0$, there exists $N \in \mathbb{N}$ so that for all $n > N, a_n > K$ (respectively, $a_n < -K$). In this case, we write

$$\lim_{n\to\infty} a_n = \infty, \text{ (respectively, } \lim_{n\to\infty} a_n = -\infty).$$

A divergent sequence may also *oscillate* between different values, e.g. $((-1)^n)$ takes only two values $+1$ and -1.

Definition A sequence (a_n) is said to be *bounded above* (respectively, *bounded below, bounded*) if the set $\{a_n, n \in \mathbb{N}\}$ is bounded above (respectively, *bounded below, bounded*).

For example $(1/n)$ and $((-1)^n)$ are both bounded.

Theorem 2.1.2. *If a sequence (a_n) is convergent, then it is also bounded.*

Proof. We need to find $K > 0$ such that $|a_n| \leq K$ for all $n \in \mathbb{N}$. We know (a_n) converges to some $l \in \mathbb{R}$, so given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that if $n > N$, then $|a_n - l| < \epsilon$. By the triangle inequality, if $n > N$

$$|a_n| = |(a_n - l) + l| \leq |a_n - l| + |l| < \epsilon + |l|.$$

So we can take $K = \epsilon + |l|$, provided $n > N$. We need a K that works for all $n \in \mathbb{N}$, so now suppose that $n \leq N$. We then have

$$|a_n| \leq \max(\{|a_1|, |a_2|, \ldots, |a_N|\}).$$

If we combine together the two pieces of our argument we see that we need

$$K = \max(\{|a_1|, |a_2|, \ldots, |a_N|, \epsilon + |l|\}),$$

and that completes the proof. □

The converse to Theorem 2.1.2 is not true; for a counter–example, consider the sequence whose nth term is $(-1)^n$; it is bounded, but not convergent.

2.2 The Algebra of Limits

Consider the sequence whose nth term is $a_n = \dfrac{2n + 3n^2}{n^2 + 4}$. If you do some numerical experiments, you might conjecture that the limit is 3. How would we prove this? Divide every term in the numerator and denominator of the fraction by the highest power of n that occurs. This is n^2 and we get:

$$a_n = \frac{\frac{2}{n} + 3}{1 + \frac{4}{n^2}}.$$

Now we know that $\lim_{n\to\infty} \frac{1}{n} = 0, \lim_{n\to\infty} 3 = 3, \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} 2 = 2$. We could indeed argue that $\lim_{n\to\infty} a_n = 3$ if we could justify writing

$$\lim_{n\to\infty} a_n = \frac{2 \lim_{n\to\infty} \frac{1}{n} + 3}{1 + 4 \lim_{n\to\infty} \frac{1}{n} \lim_{n\to\infty} \frac{1}{n}}.$$

It turns out that this sort of reasoning is indeed justified, and the general result that we need is given in the next theorem – which is often known as *the algebra of limits.*

Theorem 2.2.1 (The Algebra of Limits)**.** *Suppose that* (a_n) *and* (b_n) *are convergent sequences with* $\lim_{n\to\infty} a_n = l$ *and* $\lim_{n\to\infty} b_n = m$ *then*

1. *The sequence whose* n*th term is* $a_n + b_n$ *converges to* $l + m$.

2. *The sequence whose* n*th term is* $a_n b_n$ *converges to* lm.

3. *If* c *is any real number then the sequence whose* n*th term is* ca_n *converges to* cl.

4. *If* $b_n \neq 0$ *for all* n *and also* $m \neq 0$ *then the sequence whose* n*th term is* $\dfrac{a_n}{b_n}$ *converges to* $\dfrac{l}{m}$.

Proof. 1. This is similar to that of Theorem 2.1.1: given any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - l| < \frac{\epsilon}{2}$, and there also exists $M \in \mathbb{N}$ such that if $n > M$, then $|b_n - m| < \frac{\epsilon}{2}$. Now choose any $n > \max\{M, N\}$ and apply the triangle inequality to see that

$$\begin{aligned} |(a_n + b_n) - (l + m)| &= |(a_n - l) + (b_n - m)| \\ &\leq |a_n - l| + |b_n - m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

2. Here we'll use the triangle inequality in a familiar way - but we'll also need to appeal to Theorem 2.1.2. First we have

$$\begin{aligned} |a_n b_n - lm| &= |a_n b_n - l b_n + l b_n - lm| \\ &\leq |b_n(a_n - l)| + |l(b_n - m)| \\ &= |b_n||a_n - l| + |l||b_n - m|, \ldots (*) \end{aligned}$$

At this stage we'll assume that $l \neq 0$ and worry about what happens when $l = 0$ later on. Now the sequence (b_n) is convergent and so by Theorem 2.1.2 is bounded. Hence there exists a real number $K > 0$ such that $|b_n| \leq K$ for all n. So we can go back to (*) and write

$$|a_n b_n - lm| \leq K|a_n - l| + |l||b_n - m| \ldots (**)$$

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n > N$, $|a_n - l| < \dfrac{\epsilon}{2K}$, and there exists $M \in \mathbb{N}$ such that $|b_n - m| < \dfrac{\epsilon}{2|l|}$ whenever $n > M$.

From (**) we then get for $n > \max\{M, N\}$,

$$\begin{aligned} |a_n b_n - lm| &< K.\frac{\epsilon}{2K} + |l|.\frac{\epsilon}{2|l|} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

That proves the theorem in the case where $l \neq 0$. If $l = 0$ then just go back to (*) and use the fact that given $\epsilon > 0$ there exists $P \in \mathbb{N}$ such that if $n > P$, $|a_n| < \dfrac{\epsilon}{K}$.

3. This follows from (2) by taking (b_n) to be the constant sequence whose nth term is c.

4. First we'll show that $\lim_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{m}$. Consider

$$\begin{aligned} \left|\frac{1}{b_n} - \frac{1}{m}\right| &= \left|\frac{m - b_n}{m b_n}\right| \\ &= \frac{1}{|b_n|}.\frac{1}{|m|}.|b_n - m| \ldots (\dagger) \end{aligned}$$

For large enough n we can make $|b_n - m|$ as small as we like and $\frac{1}{|m|}$ is constant and so presents no problems. The problem term in (†) is $\dfrac{1}{|b_n|}$ so lets focus on that. To deal with this we'll need to be clever and we'll choose $\epsilon < \dfrac{|m|}{2}$. As we continue the argument, you'll see why this is a good idea. Given such an ϵ we can as usual find $N \in \mathbb{N}$ such that if $n > N$ then

$$|b_n - m| < \epsilon < \frac{|m|}{2}.$$

So by Theorem 1.3.1, we have

$$|m| - |b_n| < \frac{|m|}{2},$$

and so, $|b_n| > \dfrac{|m|}{2}$. Hence

$$\frac{1}{|b_n|} < \frac{2}{|m|}.$$

We can then use the same argument as in the proof of Theorem 2.1.2 to see that the sequence whose nth term is $\dfrac{1}{|b_n|}$ is bounded with $K = \max\left(\frac{1}{|b_1|}, \frac{1}{|b_2|}, \ldots, \frac{1}{|b_N|}, \frac{2}{|m|}\right)$.

Now lets return to (†). With ϵ as chosen, we see that for $n > N$ we have

$$\left|\frac{1}{b_n} - \frac{1}{m}\right| < \frac{K}{|m|}\epsilon,$$

and that will suffice to establish that $\lim_{n\to\infty} \dfrac{1}{b_n} = \dfrac{1}{m}$. Finally to show the general result claimed in the theorem we just write $\dfrac{a_n}{b_n} = a_n.\dfrac{1}{b_n}$ and use the result of (2) that was proved above.

□

Now that we've proved Theorem 2.2.1 you should go back to the sequence we considered at the beginning of the section, and convince yourself that every step can be justified to prove that the limit is 3.

Example 2.3 Find $\lim\limits_{n\to\infty} \dfrac{2n - 7n^3}{6n^2 + 11n^3}$

The trick in problems like this is to divide top and bottom by the highest power of n (in this case n^3), and then use the algebra of limits systematically:

$$\begin{aligned}
\lim_{n\to\infty} \frac{2n - 7n^3}{6n^2 + 11n^3} &= \lim_{n\to\infty} \frac{2n/n^3 - 7n^3/n^3}{6n^2/n^3 + 11n^3/n^3} \\
&= \lim_{n\to\infty} \frac{2/n^2 - 7}{6/n + 11n} = \frac{0 - 7}{0 + 11} = -7/11.
\end{aligned}$$

Another very useful result for finding limits is the *sandwich rule*:

Theorem 2.2.2 (Sandwich Rule)**.** *Suppose we are given three sequences $(a_n), (b_n)$ and (c_n), so that for all $n \in \mathbb{N}$ we have $a_n \leq b_n \leq c_n$. If (a_n) and (c_n) both converge to the same limit l, then (b_n) also converges to l.*

Proof. We have to show that given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that for all $n > N, |b_n - l| < \epsilon$, i.e. both $b_n - l < \epsilon$ and $l - b_n < \epsilon$. But there exists $M_1 \in \mathbb{N}$ so that for all $n > M_1$,

$$b_n - l \leq c_n - l \leq |c_n - l| < \epsilon;$$

and there exists $M_2 \in \mathbb{N}$ so that for all $n > M_2$,

$$l - b_n \leq l - a_n \leq |a_n - l| < \epsilon.$$

The result follows by taking $N = \max\{M_1, M_2\}$. $\square$

Example 2.4 Find $\lim_{n\to\infty} \dfrac{\cos(n)}{n}$.

For all $n \in \mathbb{N}$, we have $-1 \leq \cos n \leq 1$, and so

$$-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by the sandwich rule, we have $\lim_{n\to\infty} \dfrac{\cos(n)}{n} = 0$.

2.3 Bounded Monotonic Sequences

In this section, we study an important class of sequences.

Definition A sequence (a_n) is *monotonic increasing if* $a_{n+1} \geq a_n$ *for all* $n \in \mathbb{N}$*, and*[1] *it is monotonic decreasing if* $a_{n+1} \leq a_n$ *for all* $n \in \mathbb{N}$. Finally we say that sequence is *monotone* if it is either monotonic increasing or decreasing.

It is easy to see that (a_n) is monotonic increasing if and only if $(-a_n)$ is monotonic decreasing. An example of a monotonic decreasing sequence is $a_n = \frac{1}{n}$ while $a_n = 1 - \frac{1}{n}$ is monotonic increasing.

Now it certainly isn't true that every monotone sequence converges, e.g. think of $a_n = n$. But suppose a sequence is both monotone increasing and bounded above. Then on the one hand we are told that our sequence is steadily increasing in value, but on the other hand, we have imposed a ceiling on it that it cannot exceed. So where can it go too except to the ceiling? The next result puts this intuition into precise mathematical form.

Theorem 2.3.1.

1. *If the sequence* (a_n) *is bounded above and monotonic increasing then it is convergent and* $\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}}(a_n)$.

2. *If the sequence* (a_n) *is bounded below and monotonic decreasing then it is convergent and* $\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}}(a_n)$.

[1] We sometimes say it is *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

Proof. 1. Since (a_n) is bounded above, $\alpha = \sup_{n\in\mathbb{N}}(a_n)$ exists by the completeness property. By Proposition 1.4.2, given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $a_N > \alpha - \epsilon$. But (a_n) is monotonic increasing, so for all $n > N$ we have $a_n \geq a_{N+1} \geq a_N > \alpha - \epsilon$. It follows that

$$|\alpha - a_n| = \alpha - a_n < \epsilon, \text{ for all } n \geq N,$$

and the result follows.

2. This is Problem 35.

$\square$

Corollary 2.3.2. *1. If the sequence (a_n) is monotonic increasing, then either it converges or it diverges to $+\infty$.*

2. If the sequence (a_n) is monotonic decreasing, then either it converges or it diverges to $-\infty$.

Proof. We only prove (2) as (1) is so similar. Suppose that (a_n) is monotonic decreasing. Then either it is bounded below or it isn't. If it is bounded below then it converges by Theorem 2.3.1 (2). If it isn't bounded below then given any $K < 0$ we can find $N \in \mathbb{N}$ such that $a_N < K$, for otherwise K would be a lower bound. But then since the sequence is monotonic decreasing we have $a_n < K$ for all $n \geq N$ and so the sequence diverges to $-\infty$. $\square$

Example. (Exam Style Question) Consider the sequence (a_n) given by

$$a_1 = 0, \quad a_{n+1} = \frac{3a_n + 1}{a_n + 3} \text{ for all } n > 1, \tag{2.3.1}$$

(a) Use induction to show that $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$.

(b) Show that (a_n) is monotonic increasing.

(c) Explain why $\lim_{n\to\infty} a_n$ exists, and find its value.

Solution.

(a) It's clearly true for $n = 1$. Assume true for some value of n. Then

$$a_{n+1} \geq \frac{3.0 + 1}{1 + 3} = \frac{1}{4} > 0,$$

$$\begin{aligned} a_{n+1} - 1 &= \frac{3a_n + 1}{a_n + 3} - 1 \\ &= \frac{2(a_n - 1)}{a_n + 3} \\ &\leq \frac{2 - 2}{3} = 0, \end{aligned}$$

so $0 \leq a_{n+1} \leq 1$, and the required result is true, by induction.

(b) For all $n \in \mathbb{N}$,

$$\begin{aligned} a_{n+1} - a_n &= \frac{3a_n + 1}{a_n + 3} - a_n \\ &= \frac{3a_n + 1 - a_n^2 - 3a_n}{a_n + 3} \\ &= \frac{1 - a_n^2}{a_n + 3} \geq 0 \text{ (by (a))}, \end{aligned}$$

and so $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, i.e. (a_n) is monotonic increasing.

(c) By (a) the sequence is bounded above (by 1), and by (b) it is monotonic increasing. So it converges to a limit by Theorem 2.3.1 (1). Let $\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. Now take limits on both sides of the general formula in (2.3.1) and use algebra of limits to get

$$\alpha = \frac{3\alpha + 1}{\alpha + 3}.$$

From this, we get $\alpha^2 = 1$, i.e. $\alpha = \pm 1$. but $a_1 = 0$ and the sequence is monotonic increasing. So we must have $\lim_{n\to\infty} a_n = 1$.

Example 2.5 (The Golden Section as a Limit)

We'll construct a sequence (a_n) by *recursion*, so that a_{n+1} is not given explicitly by a known formula, but implicitly through the value of a_n. This doesn't work unless we have a starting point and so we define (a_n) by:

$$a_1 = 1 \text{ and } a_{n+1} = \sqrt{1 + a_n} \text{ for } n \geq 1. \tag{2.3.2}$$

Let's calculate the first few terms. We have

$$
\begin{array}{lcrcl}
a_2 & = & \sqrt{1+1} & = & 1.4142136\ldots \\
a_3 & = & \sqrt{1+\sqrt{2}} & = & 1.553774\ldots \\
a_4 & = & \sqrt{1+\sqrt{1+\sqrt{2}}} & = & 1.5980532\ldots \\
a_5 & = & \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} & = & 1.6109997\ldots \\
a_6 & = & \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}}} & = & 1.658588\ldots
\end{array}
$$

It certainly looks like (a_n) is increasing and bounded above. How do we prove this? Let's look at the bounded problem first.

Bounded. From the calculations we've done it certainly looks like 2 will be an upper bound. There's no good reason why it should be the sup but finding that isn't our concern...yet. Let's use a proof by contradiction and suppose that there exists a number N such that $a_n \leq 2$ for all $1 \leq n \leq N$ but $a_{N+1} > 2$. From our calculations we know that if it exists then $N > 6$. Now by (2.3.2), $\sqrt{1 + a_N} > 2$ and squaring this yields

$$1 + a_N > 4,$$

i.e. $a_N > 3$. That's a contradiction, and so we can assert that our sequence is bounded above.

Monotone. Squaring the general recursive formula in (2.3.2) we get for all $n \geq 1$,

$$a_{n+1}^2 = 1 + a_n,$$

and for all $n \geq 2$,

$$a_n^2 = 1 + a_{n-1}.$$

Subtracting the second equation from the first yields

$$a_{n+1}^2 - a_n^2 = a_n - a_{n-1},$$

$$\text{i.e.} \quad (a_{n+1} + a_n)(a_{n+1} - a_n) = a_n - a_{n-1},$$

and so, noting that $a_n > 1$ for all $n \geq 2$ (Why is this true?), we get

$$a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n}.$$

Working backwards we get

$$a_n - a_{n-1} = \frac{a_{n-1} - a_{n-2}}{a_n + a_{n-1}},$$

and continuing in this manner we eventually get to

$$a_3 - a_2 = \frac{a_2 - a_1}{a_3 + a_2}.$$

Combining these all together we find that

$$\begin{aligned} a_{n+1} - a_n &= \frac{a_2 - a_1}{(a_{n+1} + a_n)(a_n + a_{n-1}) \cdots (a_3 + a_2)} \\ &= \frac{\sqrt{2} - 1}{(a_{n+1} + a_n)(a_n + a_{n-1}) \cdots (a_3 + a_2)} > 0, \end{aligned}$$

as $\sqrt{2} > 1$ and the bottom line of the fraction is a positive number. This shows that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, and so (a_n) is monotonic increasing.

Limit. As the sequence is bounded above and monotonic increasing we know that it converges by Theorem 2.3.1. Let $l = \sup_{n\in\mathbb{N}}(a_n) = \lim_{n\to\infty} a_n$. To find l we'll first square both sides of (2.3.2) to get

$$a_{n+1}^2 = 1 + a_n,$$

and then take limits of both sides

$$\lim_{n\to\infty} a_{n+1}^2 = \lim_{n\to\infty} (1 + a_n).$$

Now apply the algebra of limits (Theorem 4.3.1) and we obtain a quadratic equation in l:

$$\begin{aligned} l^2 &= l + 1, \\ \text{i.e.} \quad l^2 - l - 1 &= 0. \end{aligned}$$

This equation has two solutions - the *golden section* $\phi = \dfrac{1 + \sqrt{5}}{2}$ and $1 - \phi$. In our case, since the limit is the supremum, and every term of the sequence is a positive number, we must have $l > 0$ and so $l = \phi$. The golden section, or golden ratio, has fascinated many thinkers since antiquity. The author, Mario Livio, of a book[2] about this number writes:

"Some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties. But the fascination with the Golden Ratio is not confined just to mathematicians. Biologists, artists, musicians, historians, architects, psychologists, and even

[2] "The Golden Ratio: The Story of Phi, the World's Most Astonishing Number"

mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics."

Example 2.6 (e as a limit.)

One of the most natural definitions of the number e, which is, among other things, the base of natural logarithms, is

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Here we prove that the limit exists, and as a bonus, we find that $2 \leq e \leq 3$. More work is needed to show that e is irrational, and that

$$e = 2.7182818284590452353602874713526624977572470936999595749669...$$

Once again, we will obtain the result by using Theorem 2.3.1, and will show that the sequence whose nth term is $(1+1/n)^n$ is monotonic increasing and bounded above.

Monotone Increasing. To prove this we'll use the theorem of the means, i.e. if $a_1, a_2, \ldots, a_n \geq 0$ then

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Apply this result with $a_1 = a_2 = \cdots = a_{n-1} = 1 + \frac{1}{n-1}$ and $a_n = 1$. Then the geometric mean is $\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{n}}$ and the arithmetic mean is

$$\frac{(n-1)\left(1+\frac{1}{n-1}\right)+1}{n} = 1 + \frac{1}{n}.$$

So the theorem of the means tells us that for all $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n-1}\right)^{\frac{n-1}{n}} \leq 1 + \frac{1}{n},$$

and raising both sides to the power n gives

$$\left(1+\frac{1}{n-1}\right)^{n-1} \leq \left(1+\frac{1}{n}\right)^n,$$

and so our sequence is monotonic increasing.

Bounded. To prove this we first use the binomial theorem to expand

$$\left(1+\frac{1}{n}\right)^n = 1 + \frac{1}{n}.n + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{1}{n^n}.$$

From here we have the inequality $\left(1+\frac{1}{n}\right)^n > 2$, for all $n \in \mathbb{N}$, which we'll return to later.

Using a little bit of algebra, we can rewrite the binomial expansion to get

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= 1+\frac{1}{1!}+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!} \\
&+ \cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)\frac{1}{n!} \\
&\le 1+\sum_{r=1}^{n}\frac{1}{r!}
\end{aligned}$$

Since $r! \ge 2^{r-1}$ for all $r \in \mathbb{N}$, we have (after summing the geometric progression):

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &\le 1+\sum_{r=1}^{n}\frac{1}{2^{r-1}} \\
&= 1+2(1-1/2^n) \le 3.
\end{aligned}$$

This tells us that the sequence is bounded above, and so e exists, as promised. We have also obtained the inequalities:

$$2 \le \left(1+\frac{1}{n}\right)^n \le 3,$$

and so taking the limit as $n \to \infty$, and using Problem 31(b), we deduce that $2 \le e \le 3$.

In Semester 2, you will show that $e = \sum_{n=0}^{\infty}\frac{1}{n!}$, and more generally, for all $x \in \mathbb{R}$, $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

2.4 Subsequences

By considering some, but not all, of the terms in a sequence, we get a subsequence.

Definition A sequence (y_n) is a *subsequence* of a sequence (x_n) if there is a strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ so that $y_n = x_{\sigma(n)}$ for all $n \in \mathbb{N}$.

Equivalently a subsequence of $(x_n, n \in \mathbb{N})$ is a sequence of the form $(x_{n_r}, r \in \mathbb{N})$, i.e. $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$, where $n_1 < n_2 < n_3 < \cdots$.

For example, $(1/2n), (1/3n), (1/(5n-3))$ are all subsequences of $(1/n)$. In fact $(1/an+b)$ is a subsequence of $(1/n)$ for all $a \in \mathbb{N}, b \in \mathbb{Z}$ with $a+b > 0$.

Proposition 2.4.1. *If the sequence (a_n) converges to some limit l, then every subsequence of (a_n) also converges to l.*

Proof. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that if $n > N$, then $|a_n - l| < \epsilon$. Let (a_{n_r}) be a subsequence of (a_n) and choose $K \in \mathbb{N}$ so that $n_K > N$. Then for all $r > K$, $|a_{n_r} - l| < \epsilon$, and we are done. $\square$

It can be more interesting to seek convergent subsequences of divergent sequences, e.g. the sequence whose nth term is $(-1)^n$, has two convergent subsequences, obtained by taking odd and even terms, respectively. Another way in which subsequences can be important, is to provide tools to prove that a sequence converges; sometimes it turns out that the best way to do this is to first find a subsequence converging to l, say, and then show that the whole sequence converges to l. We'll see an example of this in the next section (Theorem 2.5.1), where we will also use the important Bolzano–Weierstrass theorem, which we will prove after the next result, which is a key step on the way.

Theorem 2.4.2. *Every sequence has a monotone subsequence.*

Proof. Let (a_n) be a sequence and define

$$C = \{N \in \mathbb{N}; a_m < a_N \text{ for all } m > N\}.$$

The set C is either bounded above or it isn't.

Suppose that C is bounded above. Then C is finite. We choose $n_1 \in \mathbb{N}$ as follows:

$$\begin{gathered}\text{If } C = \emptyset, \text{ then } n_1 = 1.\\ \text{If } C \neq \emptyset, \text{ then } n_1 = \max(C) + 1.\end{gathered}$$

In either case, $n \notin C$ for all $n \geq n_1$. But

$$\begin{aligned} n_1 \notin C &\Rightarrow a_{n_2} \geq a_{n_1} \text{ for some } n_2 > n_1,\\ n_2 \notin C &\Rightarrow a_{n_3} \geq a_{n_2} \text{ for some } n_3 > n_2,\end{aligned}$$

and we can use induction to complete the proof that $(a_{n_j}, j \in \mathbb{N})$ is monotonic increasing.

Suppose that C is unbounded. Then we can find an infinite sequence $(n_1, n_2, n_3, \ldots)$ in C, with $n_1 < n_2 < n_3 < \cdots$

$$\begin{aligned} n_1 \in C &\Rightarrow a_m < a_{n_1} \text{ for all } m > n_1 \Rightarrow a_{n_2} < a_{n_1},\\ n_2 \in C &\Rightarrow a_m < a_{n_2} \text{ for all } m > n_2 \Rightarrow a_{n_3} < a_{n_2},\end{aligned}$$

and we again use induction to complete the proof that $(a_{n_k}, k \in \mathbb{N})$ is monotonic decreasing. $\square$

We will see in the next section how important the next result is:

Theorem 2.4.3. *[Bolzano-Weierstrass] Every bounded sequence has a convergent subsequence.*

Proof. Suppose (a_n) is bounded. By Theorem 2.4.2 it has a monotone subsequence (a_{n_r}) which is itself bounded. Then (a_{n_r}) converges by Theorem 2.3.1. $\square$

2.5 Cauchy Sequences

In this last section of the chapter, we return to one of the themes of Chapter 1: the completeness of the real numbers. In fact, you met Cauchy sequences before, at the end of MAS114 semester 1.

Definition A sequence (a_n) is said to be a *Cauchy sequence* if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that for all $m, n > N$ we have $|a_m - a_n| < \epsilon$.

Notice that the definition says nothing about convergence; but it does tell us that terms of the sequence get arbitrarily close to each other if we move far enough along it. The next two results are left for you to prove for your self in the exercises (Problems 41 and 42):

- Every convergent sequence is Cauchy.
- Every Cauchy sequence is bounded.

The next result is very important. It is equivalent to the completeness property of the real numbers. A proof of that fact can be found on the course website.[3]

Theorem 2.5.1. *If (a_n) is a Cauchy sequence in $\mathbb{R}$, then it converges to a limit in $\mathbb{R}$.*

Proof. By Problem 41, (a_n) is bounded, hence by the Bolzano-Weierstrass theorem it has a subsequence (a_{n_r}) (say) converging to some $l \in \mathbb{R}$. We will prove that (a_n) converges to l. In fact given any $\epsilon > 0$, there exists $M_1 \in \mathbb{N}$ so that for all $m, n > M_1$ we have $|a_m - a_n| < \epsilon/2$, and there exists $M_2 \in \mathbb{N}$

[3]Look for the file – Characterising Completeness of the Real Number Line.

so that $n_r > M_2 \Rightarrow |a_{n_r} - l| < \epsilon/2$. Hence for all $n > M_1$ and choosing r so that $n_r > \max\{M_1, M_2\}$, we have

$$\begin{aligned} |a_n - l| &\leq |a_n - a_{n_r}| + |a_{n_r} - l| \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

and the result is proved. □

Definition A non–empty subset $A \subseteq \mathbb{R}$ is said to be *complete* if every Cauchy sequence taking values in A converges to a limit in A.

Theorem 2.5.1 tells us that $\mathbb{R}$ is complete; but $\mathbb{Q}$ is not, for consider the sequence $(1, 1.4.1.41, 1.414, 1.4142, \ldots)$ of rational approximations to $\sqrt{2}$. It is Cauchy (as it converges to $\sqrt{2}$), and each term of the sequence lies in $\mathbb{Q}$, but the limit does not. One way of constructing the real number line is to take the union of the rational numbers with all limits of Cauchy sequences of rational numbers. Of course this requires quite a lot of work to ensure that it delivers the goods.