Chapter 3

Lebesgue Integration

3.1 Introduction

The concept of integration as a technique that both acts as an inverse to the operation of differentiation and also computes areas under curves goes back to the origin of the calculus and the work of Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716). It was Leibniz who introduced the $\int \cdots dx$ notation. The first rigorous attempt to understand integration as a limiting operation within the spirit of analysis was due to Bernhard Riemann (1826-1866). The approach to *Riemann integration* that is usually taught (as in MAS207) was developed by Jean-Gaston Darboux (1842-1917). At the time it was developed, this theory seemed to be all that was needed but as the 19th century drew to a close, some problems appeared:

- One of the main tasks of integration is to recover a function f from its derivative f'. But some functions were discovered for which f' was bounded but not Riemann integrable.
- Suppose (f_n) is a sequence of functions converging pointwise to f. The Riemann integral could not be used to find conditions for which

$$\int f(x)dx = \lim_{n\to\infty} \int f_n(x)dx.$$

- Riemann integration was limited to computing integrals over $\mathbb{R}^n$ with respect to Lebesgue measure. Although it was not yet apparent, the emerging theory of probability would require the calculation of expectations of random variables X: $\mathbb{E}(X) = \int_\Omega X(\omega)dP(\omega)$.

In this chapter, we'll study Lebesgue's powerful techniques which allow us to investigate $\int_S f(x)dm(x)$ where $f : S \to \mathbb{R}$ is a "suitable" measurable function defined on a measure space (S, Σ, m).[1] If we take m to be Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ we recover the familiar integral $\int_{\mathbb{R}} f(x)dx$ but we will now be able to integrate many more functions (at least in principle) than Riemann and Darboux. If we take X to be a random variable on a probability space, we get its expectation $\mathbb{E}(X)$.

Notation. For simplicity we usually write $\int_S f dm$ instead of $\int_S f(x)dm(x)$. Note that many authors use $\int_S f(x)m(dx)$. To simplify even further we'll sometimes write $I(f) = \int_S f dm$.

We'll present the construction of the Lebesgue integral in four steps: Step 1: Indicator functions, Step 2: Simple Functions, Step 3: Non-negative measurable functions, Step 4: Integrable functions.

3.2 The Lebesgue Integral for Simple Functions

Step 1. **Indicator Functions**

This is very easy and yet it is very important:
If $f = \mathbf{1}_A$ where $A \in \Sigma$

$$\int_S \mathbf{1}_A dm = m(A). \tag{3.2.1}$$

e.g. In a probability space we get $\mathbb{E}(\mathbf{1}_A) = P(A)$.

Step 2. **Simple Functions**

Let $f = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$ be a non-negative simple function so that $c_i \geq 0$ for all $1 \leq i \leq n$. We extend (3.2.1) by linearity, i.e. we define

$$\int_S f dm = \sum_{i=1}^n c_i m(A_i), \tag{3.2.2}$$

and note that $\int_S f dm \in [0, \infty]$.

Theorem 3.2.1 *If f and g are non-negative simple functions and $\alpha, \beta \geq 0$ then*

1. *("Linearity") $\int_S (\alpha f + \beta g)dm = \alpha \int_S f dm + \beta \int_S g dm$,*
2. *(Monotonicity) If $f \leq g$ then $\int_S f dm \leq \int_S g dm$.*

[1] We may also integrate extended measurable functions, but will not develop that here.

Proof

1. Let $f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}, g = \sum_{j=1}^{m} d_j \mathbf{1}_{B_j}$. Since $\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j = S$, by the last part of Problem 10 we see that

$$f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i \cap S} = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i \cap \bigcup_{j=1}^{m} B_j} = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \mathbf{1}_{A_i \cap B_j}.$$

It follows that

$$\alpha f + \beta g = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha c_i + \beta d_j) \mathbf{1}_{A_i \cap B_j},$$

and so

$$\begin{aligned} I(\alpha f + \beta g) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha c_i + \beta d_j) m(A_i \cap B_j) \\ &= \alpha \sum_{i=1}^{n} c_i m\left(A_i \cap \bigcup_{j=1}^{m} B_j\right) + \beta \sum_{j=1}^{m} d_j m\left(\bigcup_{i=1}^{n} A_i \cap B_j\right) \\ &= \alpha \sum_{i=1}^{n} c_i m(A_i) + \beta \sum_{j=1}^{m} d_j m(B_j) \\ &= \alpha I(f) + \beta I(g). \end{aligned}$$

2. By (1), $I(g) = I(f) + I(g - f)$ but $g - f$ is a non-negative simple function and so $I(g - f) \geq 0$. The result follows. □

Notation. If $A \in \Sigma$, whenever $\int_S f dm$ makes sense for some "reasonable" measurable function $f : S \to \mathbb{R}$ we define:

$$I_A(f) = \int_A f dm = \int_S \mathbf{1}_A f dm.$$

Of course there is no guarantee that $I_A(f)$ makes sense and this needs checking at each stage. In Problem 23, you can check that it makes sense when f is non-negative and simple.

3.3 The Lebesgue Integral for Non-negative Measurable Functions

We haven't done any analysis yet and at some stage we surely need to take some sort of limit! If f is measurable and non-negative, it may seem attractive to try to take advantage of Theorem 2.4.1 and define "$\int_S f dm =$

$\lim_{n\to\infty}\int_S s_n dm$". But there are many different choices of simple functions that we could take to make an approximating sequence, and this would make the limiting integral depend on that choice, which is undesirable. Instead Lebesgue used the weaker notion of the supremum to "approximate f from below" as follows:

Step 3. **Non-negative measurable functions**

$$\int_S f dm = \sup\left\{\int_S s dm, s \text{ simple}, 0 \leq s \leq f\right\}. \tag{3.3.3}$$

With this definition, $\int_S f dm \in [0, \infty]$.

The use of the sup makes it harder to prove key properties and we'll have to postpone a full proof of linearity until the next section when we have some more powerful tools. Here are some simple properties that can be proved fairly easily.

Theorem 3.3.1 *If $f, g : S \to \mathbb{R}$ are non-negative measurable functions,*

1. *(Monotonicity) If $f \leq g$ then $\int_S f dm \leq \int_S g dm$.*
2. *$I(\alpha f) = \alpha I(f)$ for all $\alpha > 0$,*
3. *If $A, B \in \Sigma$ with $A \subseteq B$ then $I_A(f) \leq I_B(f)$,*
4. *If $A \in \Sigma$ with $m(A) = 0$ then $I_A(f) = 0$.*

Proof.

$$\begin{aligned}
(1) \quad \int_S f dm &= \sup\left\{\int_S s dm, s \text{ simple}, 0 \leq s \leq f\right\} \\
&\leq \sup\left\{\int_S s dm, s \text{ simple}, 0 \leq s \leq g\right\} \\
&= \int_S g dm
\end{aligned}$$

(2) to (4) are Problem 24. □

Lemma 3.3.1 *[Markov's inequality] If $f : S \to \mathbb{R}$ is a non-negative measurable function and $c > 0$.*

$$m(\{x \in S; f(x) \geq c\}) \leq \frac{1}{c}\int_S f dm$$

Proof. Let $E = \{x \in S; f(x) \geq c\}$. Note that $E = f^{-1}([c, \infty)) \in \Sigma$ as f is measurable (see Theorem 2.2.1 (ii)). By Theorem 3.3.1 (3) and (1),

$$\begin{aligned} \int_S f dm &\geq \int_E f dm \\ &\geq \int_E c dm \\ &= cm(E), \end{aligned}$$

and the result follows. □

Definition. Let $f, g : S \to \mathbb{R}$ be measurable. We say that $f = g$ *almost everywhere* and write this for short as $f = g$ a.e. if $m(\{x \in S; f(x) \neq g(x)\} = 0$. In Problem 31 you can show that this gives rise to an equivalence relation on the set of all measurable functions. In probability theory, we use the terminology *almost surely* for two random variables X and Y that agree almost everywhere, and we write $X = Y$ (a.s.)

Corollary 3.3.1 *If f is a non-negative measurable function and $\int_S f dm = 0$ then $f = 0$ (a.e.)*

Proof. Let $A = \{x \in S; f(x) \neq 0\}$ and for each $n \in \mathbb{N}, A_n = \{x \in S; f(x) \geq 1/n\}$. Since $A = \bigcup_{n=1}^{\infty} A_n$, we have $m(A) \leq \sum_{n=1}^{\infty} m(A_n)$ by Theorem 1.5.2, and its sufficient to show that $m(A_n) = 0$ for all $n \in \mathbb{N}$. But by Markov's inequality $m(A_n) \leq n \int_S f dm = 0$. □

In Chapter 1 we indicated that we would be able to use integration to cook up new examples of measures. Let $f : S \to \mathbb{R}$ be non-negative and measurable and define $I_A(f) = \int_A f dm$ for $A \in \Sigma$. We have $\int_\emptyset f dm = 0$ by Theorem 3.3.1 (4). To prove that $A \to I_A(f)$ is a measure we then need only prove that it is σ-additive, i.e. that $I_A(f) = \sum_{n=1}^{\infty} I_{A_n}(f)$ whenever we have a disjoint union $A = \bigcup_{n=1}^{\infty} A_n$.

Theorem 3.3.2 *If $f : S \to \mathbb{R}$ is a non-negative measurable function, the mapping from Σ to $[0, \infty]$ given by $A \to I_A(f)$ is σ-additive.*

Proof. First assume that $f = \mathbf{1}_B$ for some $B \in \Sigma$. Then by (3.2.1)

$$\begin{aligned} I_A(f) = m(B \cap A) &= m\left(B \cap \bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} m(B \cap A_n) = \sum_{n=1}^{\infty} I_{A_n}(f), \end{aligned}$$

so the result holds in this case. You can then use linearity to show that it is true for non-negative simple functions.

Now let f be measurable and non-negative. Then by definition of the supremum, for any $\epsilon > 0$ there exists a simple function s with $0 \leq s \leq f$ so that $I_A(f) \leq I_A(s) + \epsilon$. The result holds for simple functions and so by monotonicity we have

$$I_A(s) = \sum_{n=1}^{\infty} I_{A_n}(s) \leq \sum_{n=1}^{\infty} I_{A_n}(f).$$

Combining this with the earlier inequality we find that

$$I_A(f) \leq \sum_{n=1}^{\infty} I_{A_n}(f) + \epsilon.$$

But ϵ was arbitrary and so we conclude that

$$I_A(f) \leq \sum_{n=1}^{\infty} I_{A_n}(f).$$

The second half of the proof will aim to establish the opposite inequality. First let $A_1, A_2 \in \Sigma$ be disjoint. Given any $\epsilon > 0$ we can, as above, find simple functions s_1, s_2 with $0 \leq s_j \leq f$, so that $I_{A_j}(s_j) \geq I_{A_j}(f) - \epsilon/2$ for $j = 1, 2$. Let $s = s_1 \vee s_2 = \max\{s_1, s_2\}$. Then s is simple (check this), $0 \leq s \leq f$ and $s_1 \leq s, s_2 \leq s$. So by monotonicity, $I_{A_j}(s) \geq I_{A_j}(f) - \epsilon/2$ for $j = 1, 2$. Add these two inequalities to find that

$$I_{A_1}(s) + I_{A_2}(s) \geq I_{A_1}(f) + I_{A_2}(f) - \epsilon.$$

But the result is true for simple functions and so we have

$$I_{A_1 \cup A_2}(s) \geq I_{A_1}(f) + I_{A_2}(f) - \epsilon.$$

By the definition (3.3.3), $I_{A_1 \cup A_2}(f) \geq I_{A_1 \cup A_2}(s)$ and so we have that

$$I_{A_1 \cup A_2}(f) \geq I_{A_1}(f) + I_{A_2}(f) - \epsilon.$$

But ϵ was arbitrary and so we conclude that

$$I_{A_1 \cup A_2}(f) \geq I_{A_1}(f) + I_{A_2}(f),$$

which is the required inequality for unions of two disjoint sets. By induction we have

$$I_{A_1 \cup A_2 \cup \cdots \cup A_n}(f) \geq \sum_{i=1}^{n} I_{A_i}(f),$$

for any $n \geq 2$. But as $A_1 \cup A_2 \cup \cdots \cup A_n \subseteq A$ we can use Theorem 3.3.1 (3) to find that

$$I_A(f) \geq \sum_{i=1}^{n} I_{A_i}(f).$$

Now take the limit as $n \to \infty$ to deduce that

$$I_A(f) \geq \sum_{i=1}^{\infty} I_{A_i}(f),$$

as was required. □

Example. The famous *Gaussian measure* on $\mathbb{R}$ is obtained in this way by taking

$$f(x) = \tfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \text{ for } x \in \mathbb{R}.$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. Equip $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with Lebesgue measure. In Chapter 2, we introduced the probability law p_X of X as a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If for all $A \in \mathcal{B}(\mathbb{R}), p_X(A) = I(f_X \mathbf{1}_A)$ for some non-negative measurable function $f_X : \mathbb{R} \to \mathbb{R}$ then f_X is called the *probability density function* or *pdf* of X. So for all $A \in \mathcal{B}(\mathbb{R})$,

$$P(X \in A) = p_X(A) = \int_A f_X(x)dx.$$

We say that X is a *standard normal* if p_X is Gaussian measure.

We present two useful corollaries to Theorem 3.3.2:

Corollary 3.3.2 *Let $f : S \to \mathbb{R}$ be a non-negative measurable function and (E_n) be a sequence of sets in Σ with $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$. Set $E = \bigcup_{n=1}^{\infty} E_n$, Then*

$$\int_E f dm = \lim_{n\to\infty} \int_{E_n} f dm.$$

Proof. This is in fact an immediate consequence of Theorems 3.3.2 and 1.5.1, but it might be helpful to spell out the proof in a little detail, so here goes: We use the "disjoint union trick", so write $A_1 = E_1, A_2 = E_2 - E_1, A_3 = E_3 - E_2, \ldots$. Then the A_ns are mutually disjoint, $\bigcup_{n=1}^{\infty} A_n = E$ and $\bigcup_{i=1}^{n} A_i = E_n$ for all $n \in \mathbb{N}$. Then by Theorem 3.3.2

$$\int_E f dm = \sum_{i=1}^{\infty} \int_{A_i} f dm$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{A_i} f dm \\
&= \lim_{n\to\infty} \int_{A_1 \cup A_2 \cup \cdots \cup A_n} f dm \\
&= \lim_{n\to\infty} \int_{E_n} f dm. \qquad \square
\end{aligned}
$$

Corollary 3.3.3 *If f and g are non-negative measurable functions and $f = g$ (a.e.) then $I(f) = I(g)$.*

Proof. Let $A_1 = \{x \in S; f(x) = g(x)\}$ and $A_2 = \{x \in S; f(x) \neq g(x)\}$. Then $A_1, A_2 \in \Sigma$ with $A_1 \cup A_2 = S, A_1 \cap A_2 = \emptyset$ and $m(A_2) = 0$. So by Theorem 3.3.1 (4), $\int_{A_2} f dm = \int_{A_2} g dm = 0$. But $\int_{A_1} f dm = \int_{A_1} g dm$ as $f = g$ on A_1 and so by Theorem 3.3.2,

$$
\begin{aligned}
\int_S f dm &= \int_{A_1} f dm + \int_{A_2} f dm \\
&= \int_{A_1} g dm + \int_{A_2} g dm = \int_S g dm. \qquad \square
\end{aligned}
$$

3.4 The Monotone Convergence Theorem

We haven't yet proved that $\int_S (f + g) dm = \int_S f dm + \int_S g dm$. Nor have we extended the integral beyond non-negative measurable functions. Before we can do either of these, we need to establish the *monotone convergence theorem*. This is the first of two important results that address one of the historical problems of integration that we mentioned in section 3.1

Let (f_n) be a sequence of non-negative measurable functions with $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ (so the sequence is *monotonic increasing*). Note that $f = \lim_{n\to\infty} f_n$ exists and is non-negative and measurable (see Theorem 2.3.5), but f may take values in $[0, \infty]$.

Theorem 3.4.1 *[The Monotone Convergence Theorem] If (f_n) is a monotone increasing sequence of non-negative measurable functions from S to $\mathbb{R}$ then*

$$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm.$$

Proof. As $f = \sup_{n\in\mathbb{N}} f_n$, by monotonicity (Theorem 3.3.1(1)), we have

$$\int_S f_1 dm \leq \int_S f_2 dm \leq \cdots \leq \int_S f dm.$$

Hence by monotonicity of the integrals, $\lim_{n\to\infty}\int_S f_n dm$ exists (as an extended real number) and

$$\lim_{n\to\infty}\int_S f_n dm \leq \int_S f dm.$$

We must now prove the reverse inequality. To simplify notation, let $a = \lim_{n\to\infty}\int_S f_n dm$. So we need to show that $a \geq \int_S f dm$. Let s be a simple function with $0 \leq s \leq f$ and choose $c \in \mathbb{R}$ with $0 < c < 1$. For each $n \in \mathbb{N}$, let $E_n = \{x \in S; f_n(x) \geq cs(x)\}$, and note that $E_n \in \Sigma$ for all $n \in \mathbb{N}$ by Proposition 2.3.3. Since (f_n) is increasing, it follows that $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$. Also we have $\bigcup_{n=1}^{\infty} E_n = S$. To verify this last identity, note that if $x \in S$ with $s(x) = 0$ then $x \in E_n$ for all $n \in \mathbb{N}$ and if $x \in S$ with $s(x) \neq 0$ then $f(x) \geq s(x) > cs(x)$ and so for some n, $f_n(x) \geq cs(x)$, as $f_n(x) \to f(x)$ as $n \to \infty$, i.e. $x \in E_n$. By Theorem 3.3.1(3) and (1), we have

$$a = \lim_{n\to\infty}\int_S f_n dm \geq \int_S f_n dm \geq \int_{E_n} f_n dm \geq \int_{E_n} csdm.$$

As this is true for all $n \in \mathbb{N}$ and (E_n) is increasing, we find that

$$a \geq \lim_{n\to\infty}\int_{E_n} csdm.$$

But by Corollary 3.3.2 and Theorem 3.3.1(2),

$$\lim_{n\to\infty}\int_{E_n} csdm = \int_S csdm = c\int_S sdm,$$

and so we deduce that

$$a \geq c\int_S sdm.$$

But $0 < c < 1$ is arbitrary so taking e.g. $c = 1 - 1/k$ with $k = 2, 3, 4, \ldots$ and letting $k \to \infty$, we find that

$$a \geq \int_S sdm.$$

But the simple function s for which $0 \leq s \leq f$ was also arbitrary, so now take the supremum over all such s and apply (3.3.3) to get

$$a \geq \int_S fdm,$$

and the proof is complete. $\square$.

Corollary 3.4.1 *Let $f : S \to \mathbb{R}$ be measurable and non-negative. There exists an increasing sequence of simple functions (s_n) converging pointwise to f so that*

$$\lim_{n\to\infty}\int_S s_n dm = \int_S f dm.$$

Proof. Apply the monotone convergence theorem to the sequence (s_n) constructed in Theorem 2.4.1 □

Theorem 3.4.2 *Let $f, g : S \to \mathbb{R}$ be measurable and non-negative. Then*

$$\int_S (f+g)dm = \int_S f dm + \int_S g dm.$$

Proof. By Theorem 2.4.1 we can find an increasing sequence of simple functions (s_n) that converges pointwise to f and an increasing sequence of simple functions (t_n) that converges pointwise to g. Hence $(s_n + t_n)$ is an increasing sequence of simple functions that converges pointwise to $f+g$. So by Theorem 3.4.1, Theorem 3.2.1(1) and then Corollary 3.4.1,

$$\begin{aligned}\int_S (f+g)dm &= \lim_{n\to\infty}\int_S (s_n+t_n)dm \\ &= \lim_{n\to\infty}\int_S s_n dm + \lim_{n\to\infty}\int_S t_n dm \\ &= \int_S f dm + \int_S g dm.\end{aligned}$$

□

Another more delicate convergence result can be obtained as a consequence of the monotone convergence theorem. We present this as a theorem, although it is always called *Fatou's lemma* in honour of the French astronomer (and mathematician) Pierre Fatou (1878-1929). We will find an important use for this result in the next section.

Theorem 3.4.3 *[Fatou's Lemma] If (f_n) is a sequence of non-negative measurable functions from S to $\mathbb{R}$ then*

$$\liminf_{n\to\infty}\int_S f_n dm \geq \int_S \liminf_{n\to\infty} f_n dm$$

Proof. Define $g_n = \inf_{k\geq n} f_k$. Then (g_n) is an increasing sequence which converges to $\liminf_{n\to\infty} f_n$. Now as $f_l \geq \inf_{k\geq n} f_k$ for all $l \geq n$, by monotonicity (Theorem 3.3.1(1)) we have that for all $l \geq n$

$$\int_S f_l dm \geq \int_S \inf_{k\geq n} f_k dm,$$

and so

$$\inf_{l \geq n} \int_S f_l dm \geq \int_S \inf_{k \geq n} f_k dm.$$

Now take limits on both sides of this last inequality and apply the monotone convergence theorem to obtain

$$\begin{aligned} \liminf_{n\to\infty} \int_S f_n dm &\geq \lim_{n\to\infty} \int_S \inf_{k\geq n} f_k dm \\ &= \int_S \lim_{n\to\infty} \inf_{k\geq n} f_k dm \\ &= \int_S \liminf_{n\to\infty} f_n dm \end{aligned}$$

□

Note that we do not require (f_n) to be a bounded sequence, so $\liminf_{n\to\infty} f_n$ should be interpreted as an extended measurable function, as discussed at the end of Chapter 2.

3.5 Lebesgue Integration Completed: Integrability and Dominated Convergence

At last we are ready for the final step in the construction of the Lebesgue integral - the extension from non-negative measurable functions to a class of measurable functions that are real-valued.

Step 4. For the final step we first take f to be an arbitrary measurable function. We define the positive and negative parts of f, which we denote as f_+ and f_- respectively by:

$$f_+(x) = \max\{f(x), 0\}, \quad f_-(x) = \max\{-f(x), 0\},$$

so both f_+ and f_- are measurable (Corollary 2.3.2) and non-negative. We have

$$f = f_+ - f_-,$$

and using Step 3, we see that we can construct both $\int_S f_+ dm$ and $\int_S f_- dm$. Provided both of these are not infinite, we define

$$\int_S f dm = \int_S f_+ dm - \int_S f_- dm.$$

With this definition, $\int_S f dm \in [-\infty, \infty]$. We say that f is *integrable* if $\int_S f dm \in (-\infty, \infty)$. Clearly f is integrable if and only if each of f_+ and f_- are. Define $|f|(x) = |f(x)|$ for all $x \in S$. Since

$$|f| = f_+ + f_-,$$

it follows that f is integrable if and only if $|f|$ is. Using this fact, the condition for integrability of f is often written

$$\int_S |f| dm < \infty.$$

We also have the useful inequality (whose proof is Problem 30(a)):

$$\left| \int_S f dm \right| \leq \int_S |f| dm. \tag{3.5.4}$$

Theorem 3.5.1 *Suppose that f and g are integrable functions from S to $\mathbb{R}$.*

1. *If $c \in \mathbb{R}$ then cf is integrable and $\int_S cf dm = c \int_S f dm$,*
2. *$f + g$ is integrable and $\int_S (f + g) dm = \int_S f dm + \int_S g dm$,*
3. *(Monotonicity) If $f \leq g$ then $\int_S f dm \leq \int_S g dm$.*

Proof. (1) and (3) are Problem 29. For (2), we may assume that both f, g are not identically 0. The fact that $f + g$ is integrable if f and g are follows from the triangle inequality (Problem 30 (b)). To show that the integral of the sum is the sum of the integrals, we first need to consider six different cases (writing $h = f + g$) (i) $f \geq 0, g \geq 0, h \geq 0$, (ii) $f \leq 0, g \leq 0, h \leq 0$, (iii) $f \geq 0, g \leq 0, h \geq 0$, (iv) $f \leq 0, g \geq 0, h \geq 0$, (v) $f \geq 0, g \leq 0, h \leq 0$, (vi) $f \leq 0, g \geq 0, h \leq 0$. Case (i) is Theorem 3.4.2. We'll just prove (iii). The others are similar. If $h = f + g$ then $f = h + (-g)$ and this reduces the problem to case (i). Indeed we then have

$$\int_S f dm = \int_S (f + g) dm + \int_S (-g) dm,$$

and so by (1)

$$\int_S (f + g) dm = \int_S f dm - \int_S (-g) dm = \int_S f dm + \int_S g dm.$$

Now write $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$, where S_i is the set of all $x \in S$ for which case (i) holds for $i = 1, 2, \ldots, 6$. These sets are disjoint and measurable and so by a slight extension of Theorem 3.3.2,[2]

$$\int_S (f+g) dm = \sum_{i=1}^{6} \int_{S_i} (f+g) dm = \sum_{i=1}^{6} \int_{S_i} f dm + \sum_{i=1}^{6} \int_{S_i} g dm = \int_S f dm + \int_S g dm,$$

[2]This works since we only need finite additivity here.

as was required. $\square$

We now present the last of our convergence theorems, the famous *Lebesgue dominated convergence theorem* - an extremely powerful tool in both the theory and applications of modern analysis:

Theorem 3.5.2 *[Lebesgue dominated convergence theorem] Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ which converges pointwise to a (measurable) function f. Suppose there is an integrable function $g : S \to \mathbb{R}$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then f is integrable and*

$$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm.$$

[Note that we don't assume that f_n is integrable. This follows immediately from the assumptions since by monotonicity (Theorem 3.3.1(1)), $\int_S |f_n| dm \leq \int_S g dm < \infty$.]

Proof. Since (f_n) converges pointwise to f, $(|f_n|)$ converges pointwise to $|f|$. By Fatou's lemma (Theorem 3.4.3) and monotonicity (Theorem 3.5.1(3)), we have

$$\begin{aligned} \int_S |f| dm &= \int_S \liminf_{n\to\infty} |f_n| dm \\ &\leq \liminf_{n\to\infty} \int_S |f_n| dm \\ &\leq \int_S g dm < \infty, \end{aligned}$$

and so f is integrable.

Also for all $n \in \mathbb{N}, g + f_n \geq 0$ so by Fatou's lemma again,

$$\int_S \liminf_{n\to\infty} (g + f_n) dm \leq \liminf_{n\to\infty} \int_S (g + f_n) dm.$$

But $\liminf_{n\to\infty}(g+f_n) = g+\lim_{n\to\infty} f_n = g+f$ and (using Theorem 3.5.1(2)) $\liminf_{n\to\infty} \int_S (g + f_n) dm = \int_S g dm + \liminf_{n\to\infty} \int_S f_n dm$. We then conclude that

$$\int_S f dm \leq \liminf_{n\to\infty} \int_S f_n dm \qquad \ldots \text{(i)}.$$

Repeat this argument with $g + f_n$ replaced by $g - f_n$ which is also non-negative for all $n \in \mathbb{N}$. We then find that

$$-\int_S f dm \leq \liminf_{n\to\infty} \left(-\int_S f_n dm \right) = -\limsup_{n\to\infty} \int_S f_n dm,$$

and so

$$\int_S f dm \geq \limsup_{n\to\infty} \int_S f_n dm \qquad \ldots \text{(ii)}.$$

Combining (i) and (ii) we see that

$$\limsup_{n\to\infty} \int_S f_n dm \leq \int_S f dm \leq \liminf_{n\to\infty} \int_S f_n dm \qquad \ldots \text{(iii)},$$

but we always have $\liminf_{n\to\infty} \int_S f_n dm \leq \limsup_{n\to\infty} \int_S f_n dm$

and so $\liminf_{n\to\infty} \int_S f_n dm = \limsup_{n\to\infty} \int_S f_n dm$. Then by Theorem 2.1.1 $\lim_{n\to\infty} \int_S f_n dm$ exists and from (iii), we deduce that

$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm$. □

Example. Suppose that (S, Σ, m) is a finite measure space and (f_n) is a sequence of measurable functions from S to $\mathbb{R}$ which converge pointwise to f and are bounded, i.e. there exists $K > 0$ so that $|f_n(x)| \leq K$ for all $x \in S, n \in \mathbb{N}$. Then f is integrable. To see this just take $g = K$ in the dominated convergence theorem and show that it is integrable which follows from the fact that $\int_S g dm = Km(S) < \infty$.

We will not prove the next theorem, which shows that the Lebesgue integral on $\mathbb{R}$ is at least as powerful as the Riemann one (at least when we integrate over *finite* intervals.) You can find a proof on the module website.

Theorem 3.5.3 *Let $f : [a, b] \to \mathbb{R}$ be bounded and Riemann integrable. Then it is also Lebesgue integrable and the two integrals have the same value.*

But we can integrate many more functions using Lebesgue integration that are not Riemann integrable, e.g. $\int_{[a,b]} \mathbf{1}_{\mathbb{R}-\mathbb{Q}}(x)dx = (b - a)$. Note that Theorem 3.5.3 only applies to *finite* closed intervals. We need to be careful on infinite intervals.

In the following examples we will freely use the fact (which was established in MAS207) that a bounded continuous function on $[a, b]$ is Riemann integrable. Then by Theorem 3.5.3, it is Lebesgue integrable. In particular

- $f(x) = x^n$ is integrable on $[a, b]$ for $n \in \mathbb{Z}$ if $0 \notin [a, b]$,
- $f(x) = x^n$ is integrable on $[a, b]$ for $n \in \mathbb{Z}_+$ if $0 \in [a, b]$.

Example 1. Show that $f(x) = x^{-\alpha}$ is integrable on $[1, \infty)$ for $\alpha > 1$.

Proof. For each $n \in \mathbb{N}$ define $f_n(x) = x^{-\alpha}\mathbf{1}_{[1,n]}(x)$. Then $(f_n(x))$ increases to $f(x)$ as $n \to \infty$. We have

$$\int_1^\infty f_n(x)dx = \int_1^n x^{-\alpha}dx = \frac{1}{\alpha - 1}(1 - n^{1-\alpha}).$$

By the monotone convergence theorem,

$$\int_1^\infty x^{-\alpha}dx = \frac{1}{\alpha - 1}\lim_{n\to\infty}(1 - n^{1-\alpha}) = \frac{1}{\alpha - 1}.$$

Example 2. Show that $f(x) = x^\alpha e^{-x}$ is integrable on $[0, \infty)$ for $\alpha > 0$.

We use the fact that for any $M \geq 0, \lim_{x\to\infty} x^M e^{-x} = 0$, so that given any $\epsilon > 0$ there exists $R > 0$ so that $x > R \Rightarrow x^M e^{-x} < \epsilon$ and choose M so that $M - \alpha > 1$. Now write

$$x^\alpha e^{-x} = x^\alpha e^{-x}\mathbf{1}_{[0,R]}(x) + x^\alpha e^{-x}\mathbf{1}_{(R,\infty)}(x).$$

The first term on the right hand side is clearly integrable. For the second term we use that fact that for all $x > R$,

$$x^\alpha e^{-x} = x^M e^{-x}.x^{\alpha-M} < \epsilon x^{\alpha-M},$$

and the last term on the right hand side is integrable by Example 1. So the result follows by monotonicity (Theorem 3.3.1 (1)).

As a result of Example 2, we know that the *gamma function* $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ exists for all $\alpha > 1$. It can also be extended to the case $\alpha > 0$.

Example 3. Show that $\Gamma(\alpha) = \lim\limits_{n\to\infty} \dfrac{n!n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n)}$ for $\alpha > 1$.

Let $P_n^\alpha = \dfrac{n!n^\alpha}{\alpha(\alpha+1)\cdots(\alpha+n)}$. You can check (e.g. by induction and integration by parts) that

$$\frac{P_n^\alpha}{n^\alpha} = \int_0^1 (1-t)^n t^{\alpha-1}dt.$$

Make a change of variable $x = tn$ to find that

$$P_n^\alpha = \int_0^n \left(1 - \frac{x}{n}\right)^n x^{\alpha-1}dx = \int_0^\infty \left(1 - \frac{x}{n}\right)^n x^{\alpha-1}\mathbf{1}_{[0,n]}(x)dx.$$

Now the sequence whose nth term is $\left(1 - \frac{x}{n}\right)^n x^{\alpha-1}\mathbf{1}_{[0,n]}(x)$ comprises non-negative measurable functions, and is monotonic increasing to $e^{-x}x^{\alpha-1}$ as $n \to \infty$. The result then follows by the monotone convergence theorem.

3.6 Further Topics: Fubini's Theorem and Function Spaces

3.6.1 Fubini's Theorem

Let (S_i, Σ_i, m_i) be two measure spaces and consider the product space $(S_1 \times S_2, \Sigma_1 \otimes \Sigma_2, m_1 \times m_2)$ as discussed in section 1.6. We can consider integration of measurable functions $f : S_1 \times S_2 \to \mathbb{R}$ by the procedure that we've already discussed and there is nothing new to say about the definition and properties of $\int_{S_1 \times S_2} f d(m_1 \times m_2)$ when f is either measurable and non-negative (so the integral may be an extended real number) or when f is integrable (and the integral is a real number.) However from a practical point of view we would always like to calculate a double integral by writing it as a *repeated integral* so that we first integrate with respect to m_1 and then with respect to m_2 (or vice versa). *Fubini's theorem*, which we will state without proof, tells us that we can do this provided that f is integrable with respect to the product measure. It is named in honour of the Italian mathematician Guido Fubini (1879-1943).

Theorem 3.6.1 *[Fubini's Theorem] Let f be integrable on $(S_1 \times S_2, \Sigma_1 \otimes \Sigma_2, m_1 \times m_2)$ so that $\int_{S_1 \times S_2} |f(x,y)|(m_1 \times m_2)(dx, dy) < \infty$. Then*

1. *The mapping $f(x, \cdot)$ is m_2-integrable, almost everywhere with respect to m_1,*
2. *The mapping $f(\cdot, y)$ is m_1-integrable, almost everywhere with respect to m_2,*
3. *The mapping $x \to \int_{S_2} f(x,y) m_2(dy)$ is equal almost everywhere to an integrable function on S_1,*
4. *The mapping $y \to \int_{S_1} f(x,y) m_1(dy)$ is equal almost everywhere to an integrable function on S_2,*
5. $$\begin{aligned} \int_{S_1 \times S_2} f(x,y)(m_1 \times m_2)(dx, dy) &= \int_{S_1} \left(\int_{S_2} f(x,y) m_2(dy) \right) m_1(dx) \\ &= \int_{S_2} \left(\int_{S_1} f(x,y) m_1(dx) \right) m_2(dy). \end{aligned}$$

3.6.2 Function Spaces

An important application of Lebesgue integration is to the construction of Banach spaces $L^p(S, \Sigma, m)$ of equivalence classes of real-valued[3] functions that agree a.e. and which satisfy the requirement

$$||f||_p = \left(\int_S |f|^p dm \right)^{\frac{1}{p}} < \infty,$$

where $1 \leq p < \infty$. In fact $|| \cdot ||_p$ is a norm on $L^p(S, \Sigma, m)$. When $p = 2$ we obtain a Hilbert space with inner product:

$$\langle f, g \rangle = \int_S f g dm.$$

There is also a Banach space $L^\infty(S, \Sigma, m)$ where

$$||f||_\infty = \inf\{M \geq 0; |f(x)| \leq M \text{ a.e.}\}.$$

These spaces play important roles in functional analysis and its applications, including partial differential equations, probability theory and quantum mechanics.

[3]The complex case also works and is important.