Chapter 4

Continuous Functions

4.1 Definition of Continuity: Basic Properties and Examples

Intuitively we think of functions as being “continuous”, if we can draw their graphs without ever taking our pen/pencil off the paper; so the graph contains no gaps/jumps/discontinuities. So if we think of a typical point $a \in D_f$, then as x gets closer and closer to a, we expect that $f(x)$ will get closer and closer to $f(a)$. But from the work of Chapter 3, we expect that as x gets closer and closer to a, then $f(x)$ gets closer and closer to its limit at a (if this exists). This leads to the following definition:

Definition. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $a \in D_f$, if $\lim_{x\to a} f(x)$ exists and equals $f(a)$. We say that f is *continuous on a set* $S \subseteq D_f$ if it is continuous at every point of S.

The following follows from the definition of the limit of a function, Theorem 3.3.2 (and that we now have $a \in D_f$):

Theorem 4.1.1. *For a function $f : \mathbb{R} \to \mathbb{R}$ having domain D_f, the following statements are equivalent.*

1. *f is continuous at $a \in D_f$.*

2. *Given any sequence (x_n) with $x_n \in D_f$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x_n) = f(a)$.*

3. *Given any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x \in D_f$ with $|x - a| < \delta$, we have $|f(x) - f(a)| < \epsilon$.*

Remark. Using Problem 8(b) we can rewrite Theorem 4.1.1(3) as “ Given any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x \in D_f$ with

$x \in (a - \delta, a + \delta)$, we have $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$." An open interval of the form $(a - \delta, a + \delta)$, where $\delta > 0$ is called an *open neighbourhood* of $a \in \mathbb{R}$. This reformulation of the notion of continuity in terms of open neighbourhoods is important in advanced analysis.

Example 4.1. Fix $c \in \mathbb{R}$, then the constant function $f(x) = c$ is continuous on $\mathbb{R}$. To see this, let $a \in \mathbb{R}$ be arbitrary and let (x_n) be any sequence converging to a. Then $f(x_n) = c$ for all $n \in \mathbb{N}$ and so the sequence $(f(x_n))$ clearly converges to $c = f(a)$.

Example 4.2. The linear function $f(x) = x$ is also continuous on $\mathbb{R}$; for again given any sequence (x_n) converging to a, $f(x_n) = x_n$ for all $n \in \mathbb{N}$, and so the sequence $(f(x_n))$ clearly converges to $f(a) = a$.

To build more interesting examples, we need the following:

Theorem 4.1.2. *[Algebra of Limits Revisited] Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous at $a \in D_f \cap D_g$. The following functions are also continuous at a.*

(a) $f + g$,

(b) fg,

(c) αf, *for all* $\alpha \in \mathbb{R}$,

(d) $\dfrac{f}{g}$, *provided* $g(a) \neq 0$.

Proof. This is a direct consequence of Theorem 3.2.1. □

Using algebra of limits and Examples 4.1 and 4.2, we can expand our supply of continuous functions:

- using Theorem 4.1.2(b) repeatedly, we can show that $f(x) = x^n$ is continuous on $\mathbb{R}$ for all $n \in \mathbb{N}$,
- then using Theorem 4.1.2(a) and (c), it follows that every polynomial $p(x) = a_0 + a_1 x + \cdots a_n x^n$ is continuous on $\mathbb{R}$,
- finally using Theorem 4.1.2(d) we conclude that rational functions $r(x) = p(x)/q(x)$ are continuous at every point in their domain.

The continuity on $\mathbb{R}$ of functions such as $f(x) = e^{kx}, g(x) = \sin(kx), h(x) = \cos(kx)$, for $k \in \mathbb{R}$, will be established in semester 2, using power series arguments.

We also have

Theorem 4.1.3. *Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. If g is continuous at a, $g(a) \in D_f$, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

Proof. This is for you to do in Problem 57. $\square$

With Theorem 4.1.3 and results from semester 2, we can immediately deduce continuity on $\mathbb{R}$ of functions such as $f(x) = \sin\left(\frac{2x-1}{x^2+1}\right)$.

Remark Let $C(\mathbb{R})$ denote the set of all functions that are continuous on $\mathbb{R}$. Then Theorem 4.1.2 tells us that $C(\mathbb{R})$ is a subspace (also a subring, and a subalgebra) of the space $\mathcal{F}(\mathbb{R})$ of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Consider the function $g(x) = x\sin(1/x)$, with domain $D_g = \mathbb{R} \setminus \{0\}$. Using Theorems 4.1.2 and 4.1.3, it is not hard to see that g is continuous at every point of its domain. In Problem 51, we showed that $\lim_{x\to 0} g(x)$ exists and is 0. Now define a new function $\tilde{g}$ by

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in D_g \\ 0 & \text{if } x = 0 \end{cases}$$

Then $\tilde{g}$ is continuous on the whole of $\mathbb{R}$.

More generally, given two functions f_1 and f_2 from $\mathbb{R}$ to $\mathbb{R}$, we say that f_2 is an *extension* of f_1, (and that f_1 is a *restriction* of f_2) if $D_{f_1} \subseteq D_{f_2}$ and $f_1(x) = f_2(x)$ for all $x \in D_{f_1}$. If f_1 is continuous on D_{f_1} and f_2 is continuous on D_{f_2}, we say that f_2 is a *continuous extension* of f_1. So in the example just considered $\tilde{g}$ is a continuous extension of g. On the other hand, consider the function $h(x) = \sin(1/x)$ with domain $\mathbb{R} \setminus \{0\}$. It has many extensions to the whole of $\mathbb{R}$: we could e.g. define

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

but since, as was shown in Problem 50, $\lim_{x\to 0} h(x)$ does not exist, then there are no continuous extensions of h to $\mathbb{R}$.

4.2 Discontinuity

A function $f : \mathbb{R} \to \mathbb{R}$ is said to have a *discontinuity* at $a \in D_f$ if it fails to be continuous there. In this case we say that f is *discontinuous* at a. For example, the indicator function $\mathbf{1}_{[a,b]}$ is discontinuous at a, and at b, but is continuous on $\mathbb{R} \setminus \{a, b\}$. To show that a function is discontinuous at a, it is sufficient to find a sequence (x_n) in $D_f \setminus \{a\}$, such that $\lim_{n\to\infty} x_n = a$, but $\lim_{n\to\infty} f(x_n) \neq f(a)$.

We can learn more about what happens at a discontinuity by using right and left limits. We say that f is *left continuous* at $a \in D_f$ if $\lim_{x\uparrow a} f(x) = f(a)$, and *right continuous* at $a \in D_f$ if $\lim_{x\downarrow a} f(x) = f(a)$.

Example 4.3 The function $\mathbf{1}_{[a,b]}$ is left continuous at $x = b$, and right continuous at $x = a$.

We just prove the right continuity, as the other argument is so similar. Let (x_n) be any sequence with $x_n > a$ for all $n \in \mathbb{N}$ that converges to a. Then $\lim_{n\to\infty} \mathbf{1}_{[a,b]}(x_n) = 1 = \mathbf{1}_{[a,b]}(a)$, and the result follows.

Theorem 4.2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in D_f$ if and only if it is both right and left continuous there.*

Proof. This is a direct consequence of Theorem 3.3.2. □

If f is discontinuous at a but both $\lim_{x\uparrow a} f(x)$ and $\lim_{x\downarrow a} f(x)$ exist (i.e. they are real numbers) and are unequal, we say that f has a *jump discontinuity* at a. In this case the *jump at a* is defined to be

$$J_f(a) = \lim_{x\downarrow a} f(x) - \lim_{x\uparrow a} f(x),$$

e.g. the function $\mathbf{1}_{[a,b]}$ has jump discontinuities at a and b, with $J_f(a) = 1$ and $J_f(b) = -1$.

Next we consider two fascinating examples of discontinuity, which are a little more complicated. Both of these are due to Pierre Lejeune Dirichlet (1805-59).

Example 4.4 (Dirichlet's Function). The function $\mathbf{1}_{\mathbb{Q}}$ is discontinuous at every point in $\mathbb{R}$.

For convenience, write $f = \mathbf{1}_{\mathbb{Q}}$. Then $f(x) = 1$ if x is rational, and 0 if it is irrational. First we will show that f is discontinuous at every point $a \in \mathbb{Q}$. Consider the sequence whose nth term is $a + 1/n$. Then by Theorem 1.2.2, we can find an irrational b_n such that $a < b_n < a + 1/n$ for all $n \in \mathbb{N}$. By the sandwich rule, $\lim_{n\to\infty} b_n = a$, but $\lim_{n\to\infty} f(b_n) = 0 \neq 1 = f(a)$. Now suppose that a is irrational, then $a + 1/n$ is also irrational, and so by Theorem 1.4.5 there exists a rational number c_n so that $a < c_n < a + 1/n$, for all $n \in \mathbb{N}$. The rest of the argument is left to you. You should also check that f has no jump discontinuities. This function is wild!

Example 4.5 (Dirichlet's Other Function).

Consider the function $g : \mathbb{R} \to \mathbb{R}$ with domain $D_g = [0, 1)$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \\ & \text{(where the fraction is written in its lowest terms)} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

So e.g. $g(1/2) = 1/2, g(1/3) = 1/3, g(2/3) = 1/3$ etc.

It turns out that g is continuous at every irrational number, and discontinuous at every rational number. We will prove the first of these. The second is left to you to do in Problem 64.

Fix any $n \in \mathbb{N}$, and let $S_n = \{x \in [0, 1); g(x) \geq 1/n\}$. The set S_n consists of all rational numbers in $[0, 1)$ having a denominator smaller than or equal to n, and there are only finitely many of these. Choose an arbitrary irrational number $a \in (0, 1)$, and consider any $\epsilon > 0$; then by the Archimedean property of $\mathbb{R}$ there exists $N \in \mathbb{N}$ with $N > 1/\epsilon$. Now take δ to be sufficiently small so that $(a - \delta, a + \delta) \cap S_N = \emptyset$. Then for all x such that $|x - a| < \delta$, we have $|g(x) - g(a)| = |g(x)| < 1/N < \epsilon$, and continuity at a is established.

4.3 Continuity on Intervals

So far we have dealt with limits and continuity for functions from $\mathbb{R}$ to $\mathbb{R}$ with domain D_f. From now on we will make the assumption that there is a closed interval $[a, b] \subseteq D_f$ and that f is continuous at every point of $[a, b]$. We then say that *f is continuous on* $[a, b]$.[1] Many well–known functions have this property, including polynomials, sines, cosines and exponential functions. In this section, we will only be interested in the restriction of f to $[a, b]$ which we write as $f : [a, b] \to \mathbb{R}$. It turns out, as we will see, that there are some very important and powerful theorems that can be proved in this context.

4.3.1 The Intermediate Value Theorem

Theorem 4.3.1 (The Intermediate Value Theorem)**.** *Let f be continuous on $[a, b]$ with $f(a) > 0$ and $f(b) < 0$, or $f(a) < 0$ and $f(b) > 0$. Then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We only consider the case $f(a) > 0$ and $f(b) < 0$, as the argument in the other case is so similar. We first construct a sequence $([a_n, b_n]; n \in \mathbb{Z}_+)$ of (nested) intervals satisfying the following properties:

[1] In fact we could adopt a weaker definition of continuity on $[a, b]$ – that f is continuous on (a, b), right-continuous at a and left continuous at b. All the theorems that follow will also hold in that case.

(i) $[a_n, b_n] \subset [a_{n-1}, b_{n-1}]$, for all $n \in \mathbb{N}$,

(ii) $b_n - a_n = 2^{-n}(b-a)$, for all $n \in \mathbb{N}$,

(iii) $f(a_n) > 0, f(b_n) < 0$, for all $n \in \mathbb{N}$.

To do this we proceed as follows. Take $[a_0, b_0] = [a, b]$.

Now construct the interval $[a_1, b_1]$ as follows. Let $m_1 = \dfrac{(b+a)}{2}$ and define

$$[a_1, b_1] = \begin{cases} [a, m_1] & \text{if } f(m_1) < 0 \\ [m_1, b] & \text{if } f(m_1) > 0. \end{cases}$$

(If $f(m_1) = 0$, then take $c = m_1$ and the theorem is proved.) It is easily seen that (i), (ii) and (iii) all hold when $n = 1$.

Now assume that we have constructed $[a_1, b_1], \ldots, [a_n, b_n]$ satisfying (i), (ii) and (iii). Let $m_{n+1} = \dfrac{(b_n + a_n)}{2}$ and define

$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, m_{n+1}] & \text{if } f(m_{n+1}) < 0 \\ [m_{n+1}, b_n] & \text{if } f(m_{n+1}) > 0. \end{cases}$$

(If $f(m_{n+1}) = 0$ for some $n \in \mathbb{N}$ then take $c = m_{n+1}$.)

By construction (i) and (iii) hold and for (ii) we have

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \frac{1}{2}.2^{-n}(b-a) = 2^{-(n+1)}(b-a).$$

So by induction, the required sequence of intervals is constructed. Furthermore, the sequence

(a_n) is monotonic increasing and bounded above (by b),
(b_n) is monotonic decreasing and bounded below (by a).

By Theorem 2.3.1, both sequences converge, and by algebra of limits and (ii):

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{-n}(b-a) = 0.$$

Define $c = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$. Then $c \in [a, b]$. Now using the fact that f is continuous and Problem 31 we have

$f(c) = \lim_{n\to\infty} f(a_n) \geq 0$ and $f(c) = \lim_{n\to\infty} f(b_n) \leq 0$. Hence $f(c) = 0$. As both $f(a)$ and $f(b) \neq 0$, $c \notin \{a, b\}$, i.e. $c \in (a, b)$. □

Corollary 4.3.2. *Let f be continuous on $[a, b]$ with $f(a) < f(b)$. Then for each $\gamma \in (f(a), f(b))$, there exists $c \in (a, b)$ with $f(c) = \gamma$.*

Proof. This is left for you to do as Problem 66. □

Note that Corollary 4.3.2 tells us that the image (or range) of the interval $[a, b]$ under the continuous function f contains the interval $[f(a), f(b)]$, i.e. $[f(a), f(b)] \subseteq f([a, b])$.

The next result gives a nice application of analysis to the theory of equations.

Corollary 4.3.3. *Every polynomial of odd degree has at least one real root.*

Proof. We write (with $a_m > 0$; the case $a_m < 0$ is similar),

$$\begin{aligned} p(x) &= a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \\ &= x^m \left(a_m + \frac{a_{m-1}}{x} + \cdots + \frac{a_1}{x^{m-1}} + \frac{a_0}{x^m} \right). \end{aligned}$$

Then, as was shown in Problem 54, $\lim_{x\to\infty} p(x) = \infty$ and $\lim_{x\to-\infty} p(x) = -\infty$. So from the definition of divergence, there exist $-\infty < a < b < \infty$ such that $p(a) < 0$ and $p(b) > 0$. But p is continuous on $[a, b]$ and so, by the intermediate value theorem, there exists $c \in (a, b)$ such that $p(c) = 0$. □

Of course, there is no analogue of Corollary 4.3.3 when m is even, e.g. $p(x) = x^2 + 1$ has no real roots.

4.3.2 The Boundedness Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is *bounded* on the set $S \subseteq D_f$ if there exists $K > 0$, such that $|f(x)| \leq K$ for all $x \in S$. Then the set $\{f(x), x \in S\}$ is a (non–empty) bounded set of real numbers and so, by completeness, it has a supremum, which we write as $\sup_{x\in S} f(x)$, and an infimum, which we write as $\inf_{x\in S} f(x)$. We say that f *attains its bounds* on S if there exist $a, b \in S$ so that

$$f(a) = \sup_{x\in S} f(x), f(b) = \inf_{x\in S} f(x).$$

Examples of functions having domain $\mathbb{R}$ that are bounded on $\mathbb{R}$ are $f(x) = \sin(x)$ and $f(x) = \cos(x)$. Both functions attain their bounds, e.g. $f(x) = \sin(x)$ has supremum 1 and this is attained at all points of the form $(2n + 1/2)\pi/2$, where $n \in \mathbb{Z}$; similarly the infimum -1 is attained at all points of the form $(2n - 1/2)\pi/2$, where $n \in \mathbb{Z}$. We are interested in the case where S is an interval and f is continuous. The function $f(x) = 1/x$ is continuous on the interval $(0, 1)$, but it is not bounded as it is divergent at 0. The function $f(x) = x$ is clearly bounded on $(0, 1)$, but it does not attain its bounds. When we restrict to closed intervals, we have a very nice result:

Theorem 4.3.4. *[The Boundedness Theorem] If f is continuous on $[a, b]$, then it is bounded on $[a, b]$ and it attains both of its bounds there.*

Proof. We first show that f is bounded. To do this, we'll assume that it isn't, and seek a contradiction. Let (x_n) be a sequence in $[a, b]$ such that $|f(x_n)| > n$ for each $n \in \mathbb{N}$ (to construct such a sequence, we might define $x_n = \inf\{x \in [a, b]; |f(x)| > n\}$). Now since (x_n) is a bounded sequence, it has a convergent subsequence (x_{n_k}) by the Bolzano-Weierstrass theorem (Theorem 2.4.3). Let $x = \lim_{k\to\infty} x_{n_k}$. By continuity, $f(x) = \lim_{k\to\infty} f(x_{n_k})$. Then by Theorem 2.1.2 the sequence $(f(x_{n_k}))$ is bounded, and this gives our required contradiction. Indeed, to make this precise, we have just shown that there exists $L > 0$ so that $|f(x_{n_k}))| \leq L$ for all $k \in \mathbb{N}$. But by assumption, f is unbounded, and so using the Archimedean property of the real numbers, we can find $k \in \mathbb{N}$ so that $|f(x_{n_k})| > n_k > L$, and we are done.

Having proved that f is bounded on $[a, b]$, we'll prove that it attains its least upper bound. We again argue by seeking a contradiction. Let $\beta = \sup_{x\in[a,b]} f(x)$ and suppose that $\beta > f(x)$ for all $x \in [a, b]$. Then $[a, b] \subseteq D_g$ where

$$g(x) = \frac{1}{\beta - f(x)},$$

and g is continuous on $[a, b]$ by Theorem 4.1.2. By the first part of this current theorem, g is bounded on $[a, b]$, so there exists $K > 0$ such that $g(x) \leq K$ for all $x \in [a, b]$. Taking $A = \{f(x), x \in [a, b]\}$ and $\epsilon = 1/K$ in Proposition 1.4.2, there exists $f(x) \in A$ such that $f(x) > \beta - 1/K$, i.e. $\beta - f(x) < 1/K$, and so $g(x) > K$. This gives the required contradiction. The argument for the greatest lower bound is similar, and is left for you as Problem 69. □

Be clear what Theorem 4.3.4 is telling us. It says nothing about boundedness on $\mathbb{R}$. Indeed any non-constant polynomial is unbounded on $\mathbb{R}$, but its restriction to every closed interval $[a, b]$ is bounded.

Corollary 4.3.5. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and non–constant on $[a, b] \subseteq D_f$, then there exists $m < M$ so that*

$$f([a, b]) = [m, M].$$

Proof. By Theorem 4.3.4, there exists $\gamma, \delta \in [a, b]$ so that $f(\gamma) = \inf_{x\in[a,b]} f(x)$ and $f(\delta) = \sup_{x\in[a,b]} f(x)$. We take $m = f(\gamma)$ and $M = f(\delta)$. If $x \in [a, b]$, then $f(x) \in [m, M]$ and so $f([a, b]) \subseteq [m, M]$. Conversely, by Corollary 4.3.2, given any $c \in (m, M)$ there exists $y \in (\gamma, \delta)$ (or in (δ, γ) depending on which number is smallest) so that $c = f(y)$, and it follows that $[m, M] \subseteq f((\gamma, \delta)) \subseteq f([a, b])$. □

4.3.3 Inverses

Let A and B be arbitrary sets and $f : A \to B$ be a mapping with $D_f = A$. Recall from MAS114 (semester 2) that f is

- *surjective* if $R_f = B$, i.e. for all $y \in B$ there exists $x \in A$ so that $f(x) = y$,
- *injective* if whenever $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$, then $x_1 = x_2$.
- *bijective* if it is both surjective and injective.
- *invertible* if there exists a mapping $f^{-1} : B \to A$, with $D_{f^{-1}} = B$ called the *inverse* of f, for which

$$f^{-1}(f(x)) = x, \text{ for all } x \in A \text{ and } f(f^{-1}(y)) = y, \text{ for all } y \in B.$$

In MAS114, you also proved

Proposition 4.3.6. *The mapping $f : A \to B$ is invertible if and only if it is bijective.*

Now we return to consider functions f from $\mathbb{R}$ to $\mathbb{R}$ with domain $D_f \subseteq \mathbb{R}$. We extend to such functions, some ideas that we considered for sequences in Chapter 2. We say that f is *monotonic increasing* if whenever $x, y \in D_f$ with $x < y$, we have $f(x) \leq f(y)$, *monotonic decreasing* if whenever $x, y \in D_f$ with $x < y$, we have $f(x) \geq f(y)$, *monotone* if it is either monotonic increasing or decreasing, and *strictly increasing/decreasing* when the $\leq$ or $\geq$ in the above definitions is replaced with $<$ or $>$ (respectively).

Here are some simple examples; it is easy to see that $f(x) = x$ is strictly increasing on $\mathbb{R}$, and that $f(x) = 1/x$ is strictly decreasing on $(-\infty, 0) \cup (0, \infty)$. Our key theorem considers the invertibility of monotone functions that are continuous on closed intervals:

Theorem 4.3.7. *[The Inverse Function Theorem] If $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing (respectively, strictly decreasing) on $[a, b]$, then f is invertible and f^{-1} is strictly increasing on $[f(a), f(b)]$ (respectively, strictly decreasing on $[f(b), f(a)]$) and continuous on $(f(a), f(b))$ (respectively, on $(f(b), f(a))$).*

Proof. We'll just consider the case where f is strictly increasing. The argument for f strictly decreasing is very similar. By Corollary 4.3.5, the range of f (restricted to $[a, b]$) is $[m, M]$, so $f : [a, b] \to [m, M]$ is surjective. Note that as f is increasing, $m = f(a)$ and $M = f(b)$. If $x, y \in [a, b]$ with $x < y$

then $f(x) < f(y)$, so if $x \neq y$ then $f(x) \neq f(y)$. Hence f is injective.[2] So f is bijective, and hence invertible by Proposition 4.3.6. To show that f^{-1} is strictly increasing, let $m \leq \alpha < \beta \leq M$. Then since f is surjective and increasing, there exist $a \leq x < y \leq b$ with $\alpha = f(x)$ and $\beta = f(y)$. Hence

$$f^{-1}(\beta) = f^{-1}(f(y)) = y > x = f^{-1}(f(x)) = f^{-1}(\alpha).$$

We need to establish continuity of f^{-1} at every $y_0 \in (f(a), f(b))$. So given any $\epsilon > 0$, we must find $\delta > 0$ so that $y \in (f(a), f(b))$ for which $|y_0 - y| < \delta$ implies that $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$. Let $x_0 = f^{-1}(y_0)$, then $a < x_0 < b$. Now for given $\epsilon > 0$, choose $a < x_1 < x_0 < x_2 < b$ such that $\max\{x_2 - x_0, x_0 - x_1\} < \epsilon$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then since f is strictly increasing,

$$f(a) < y_1 < y_0 < y_2 < f(b),$$

and if $y \in (y_1, y_2)$ we have $|f^{-1}(y) - f^{-1}(y_0)| < \epsilon$. Finally let $\delta = \min\{y_2 - y_0, y_0 - y_1\}$, and check that the $\epsilon - \delta$ criterion is satisfied.

□

You can in fact prove a little more in Theorem 4.3.7, namely that f^{-1} is right continuous at $f(a)$ and left continuous at $f(b)$. This is left for you to do in Problem 74.

Example 4.5. Let $f(x) = x^n$ for $n \in \mathbb{N}$. Then you can prove in Problem 73 that f is strictly monotonic increasing on every $[a, b] \subset [0, \infty)$. We already know that f is continuous, hence by Theorem 4.3.7, the inverse f^{-1} exists, and is continuous and strictly monotonic increasing on (a, b). Then the "maximal domain" of f^{-1} is $[0, \infty)$ and f^{-1} is continuous on $(0, \infty)$ (and right continuous at $x = 0$). Of course we write $f^{-1}(x) = x^{1/n}$, so in particular, Theorem 4.3.7 has given us a unified method for proving the existence of positive nth roots of any positive real number.

Example 4.6. In Chapter 5, anticipating the continuity proof in next semester, we'll prove that $f(x) = e^x$ is monotonic increasing on $\mathbb{R}$. Since the range of f is $(0, \infty)$, by a similar argument to that of Example 4.5, we can deduce that it has a continuous, monotonic increasing inverse f^{-1} with domain $(0, \infty)$. Of course, in this case $f^{-1}(x) = \log_e(x)$ (or $\ln(x)$, if you prefer) and again Theorem 4.3.7 tells us that every positive real number has a natural logarithm.

[2]We have used the contrapositive here.