Chapter 5

Additional Material for MAS451/6352: Outer Measures

These brief notes are intended as a short summary of additional reading that you are expected to do outside the lectures. They emphasise the main ideas and concepts, but you will need to work carefully through all the relevant proofs as can be found in e.g. Cohn "Measure Theory", pp. 14-21. This material is examinable. It can studied straight after Chapter 1.

5.1 Outer Measures

This section introduces a very useful technique for constructing measures. In particular, it can be used to prove that Lebesgue measure, as defined in Chapter 1, really does exist.

Let S be a set and $\mathcal{P}(S)$ be the power set of S. An *outer measure* on S is a mapping $\mu^* : \mathcal{P}(S) \to [0, \infty]$ which satisfies the following axioms:

(O1) $\mu^*(\emptyset) = 0$,

(O2) If $A \subseteq B \subseteq S$, then $\mu^*(A) \leq \mu^*(B)$,

(O3) If $(A_n, n \in \mathbb{N})$ is a sequence of sets in S then,

$$\mu^* \left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

The property (O3) is called *countable subadditivity*.

The notion of outer measure is due to Constantin Carathéodory (1873-1950).

Example. *Lebesgue Outer Measure* λ^* on $\mathbb{R}$ is defined as follows: for each $A \subseteq \mathbb{R}$, let C_A be the set of all infinite sequences of bounded open intervals $(a_n, b_n; n \in \mathbb{N})$ so that $A \subseteq \bigcup_{n\in\mathbb{N}}(a_n, b_n)$. Then

$$\lambda^*(A) = \inf_{C_A} \left\{ \sum_{n=1}^{\infty} (b_n - a_n) \right\}.$$

Make sure that you can show that λ^* really is an outer measure.

The next definition is important.

Let μ^* be an outer measure on a set S. A set $B \subseteq S$ is said to be μ^**-measurable* if

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c),$$

for all $A \subseteq S$.

We use the notation $\mathcal{M}_{\mu^*}(S)$ to denote the collection of all μ^*-measurable sets for a given outer measure μ^*. The next theorem is key, as it tells us how to obtain an "honest" measure μ from a given outer measure μ^*.

Theorem 5.1.1 *If μ^* is an outer measure defined on a set S, then*

1. *The collection $\mathcal{M}_{\mu^*}(S)$ is a σ-algebra,*
2. *The restriction of μ^* to $\mathcal{M}_{\mu^*}(S)$ is a measure on $(S, \mathcal{M}_{\mu^*}(S))$.*

It is customary to use μ to denote the measure obtained from μ^* in Theorem 5.1.1 (2).

When we apply Theorem 5.1.1 to Lebesgue outer measure λ^* we obtain a measure λ on $(\mathbb{R}, \mathcal{M}_{\mu^*}(\mathbb{R}))$. This is precisely *Lebesgue measure*, as discussed in Chapter 1. There is only one piece of the puzzle missing, which is that we previously discussed Lebesgue measure as living on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. However it can be shown (make sure you have studied the proof) that $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_{\mu^*}(\mathbb{R})$ and so there is no inconsistency; in fact, Theorem 5.1.1 has enabled us to extend Theorem 5.1.1 to a larger class of sets than Borel sets. We call $\mathcal{M}_{\mu^*}(\mathbb{R})$ the σ-algebra of *Lebesgue measurable sets* on $\mathbb{R}$.

Note. There is another way of looking at the relationship between Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and on $(\mathbb{R}, \mathcal{M}_{\mu^*}(\mathbb{R}))$ using the idea of the *completion* of a measure, see e.g. Cohn pp.35-7. That material is not examinable.