Introduction

This book is written for anyone who has an interest in mathematics. It occupies territory that lies midway between a popular science book and a traditional textbook. Its subject matter is the part of mathematics that is called *analysis*. This is a very rich branch of mathematics that is also relatively young in historical terms. It was first developed in the nineteenth century but it is still very much alive as an area of contemporary research. Analysis is typically first introduced in undergraduate mathematics degree courses as providing a "rigorous" (i.e. logically flawless) foundation to a historically older branch of mathematics called the *calculus*. Calculus is the mathematics of motion and change. It evolved from the work of Isaac Newton and Gottfried Leibniz in the seventeenth century to become one of the most important tools in applied mathematics. Now the relationship between analysis and calculus is extremely important but it is not the subject matter of this book. Indeed readers do not need to know any calculus at all to read most of it.

So what is this book about? In a sense it is about two concepts - *number* and *limit*. Analysis provides the tools for understanding what numbers really are. It helps us make sense of the infinitesimally small and the infinitely large as well as the boundless realms between them. It achieves this by means of a key concept - the limit - which is one of the most subtle and exquisitely beautiful ideas ever conceived by humanity. This book is designed to gently guide the reader through the lore of this concept so that it becomes a friend.

So who is this book for? I envisage readers as falling into one of three (not necessarily disjoint) categories:

- The curious. You may have read a popular book on mathematics by a masterful expositor such as Marcus de Sautoy or Ian Stewart. These books stimulated you and started you thinking. You'd like to go further but don't have the time or background to take a formal course - and self-study from a standard textbook is a little forbidding.

- The confused. You are at university and taking a beginner's course in analysis. You are finding it hard and are seriously lost. Maybe this book can help you find your way?

- The eager. You are still at school. Mathematics is one of your favourite subjects and you love reading about it and discovering new things. You've picked up this book in the hope of finding out more about what goes on at college/university level.

To read this book requires some mathematical background but not an awful lot. You should be able to add, subtract, multiply and divide whole numbers and fractions. You should also be able to work with school algebra at the symbolic level. So you need to be able to calculate fractions like $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ and also be able to deal with the general case $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{ad-bc}{bd}$. I'll take it for granted that you can multiply brackets to get $(x+y)(a+b) = xa+ya+xb+yb$ and also recognise identities such as the "difference of two squares " $x^2 - y^2 = (x+y)(x-y)$. Beyond this it is vital that you are willing to allow your mind to engage with extensive bouts of systematic logical thought.

As I pointed out above, you don't need to know anything about calculus to read most of this book (but anything you do know can only help.) To keep things as simple as possible, I avoid the use of set theory (at least until the end of the book) and the technique of "proof by mathematical induction," but both topics are at least briefly introduced in appendices for those who would like to become acquainted with them.

Sometimes I am asked what mathematicians really do. Of course there are as many different answers as there are many different traditions within the vast scope of modern mathematics. But an essential feature of what is sometimes called "pure" mathematics is the process of "proving theorems". A theorem is a fancy way of talking about a chunk of mathematical knowledge that can be expressed in two or three sentences and that tells you something new. A proof is the logical argument we use to convince ourselves (and colleagues, students and readers) that this new knowledge is really correct. If you pick up a mathematics book in a library it may well be that 70 to 80% of it just consists of lists of theorems and proofs - one after the other. On the other hand most expository books about mathematics that are written for a general reader will contain none of these at all. In this book you'll find a halfway house. The author's goal is to give the reader a genuine insight into how mathematicians really think and work. So you're going to meet theorems and proofs - but the development of these is going to be very gentle and easy paced. Each time there will be discussion before and after and - at least in the early part of the book, when the procedures are unfamiliar - the proofs will be spelt out in much greater detail than would be the case in a typical textbook.

So what is the book about anyway? In once sense it is the story of a quest - the long quest of the human race to understand the notion of number. In a sense there are two types of number. There are those like the whole numbers $1, 2, 3, 4, 5, 6$ etc that come in discrete chunks and there are those that we call "real numbers" that form a continuum where each successive number merges into the last and there are no gaps between them.[1] It is this second type of number that is the true domain of analysis.[2] These numbers may appear to be very familiar to us and we may think that we "understand" them. For example you all know the number that we signify by π. It originates in geometry as the universal constant you obtain when the circumference of any (idealised) circle is divided by its diameter. You may think you know this number because you can find it on a pocket calculator (mine gives it as 3.1415927) but I hope to

[1]This is an imprecise suggestive statement. If you want to perceive the truth that lies behind it then you must read the whole book.

[2]To be precise *real analysis*, which is that part of analysis which deals with real sequences, series and functions. This topic should be distinguished from *complex analysis* which studies complex sequences, series and functions and which isn't the subject of this book although we briefly touch on it in Chapter 8.

be able to convince you that your calculator is lying to you. You really don't know π at all - and neither do I. This is because the calculator only tells us part of the decimal expansion of π (with enough accuracy to be fine for most practical applications) but the "true" decimal expansion of π is *infinite*. We are only human beings with limited powers and our brains our not adapted to grasp the infinite as a whole. But mathematicians have developed a tool which enables us to gain profound insights into the infinite nature of numbers by only ever using *finite* means. This tool is called the limit and this book will help you understand how it works.

Guide for Readers

There are fourteen chapters in this book which is itself divided into two parts. Part 1 comprises Chapters 1 to 6 and Part 2 is the rest. The six chapters in Part 1 can serve as a background text for a standard first year undergraduate course in numbers, sequence and series (or in some colleges and universities, the first half of a first or second year course on introductory real analysis.) Chapters 1 and 2 introduce the different types of number that feature in most of this book: natural, prime, integer, rational and real. Chapter 3 is the bridge between number and analysis. It is devoted to the art of manipulating inequalities. Chapters 4 and 5 focus on limits of sequences and begin the study of analysis proper. Chapter 6 (which is the longest in the book) deals with infinite series.

Part 2 comprises a selection of additional interesting topics. In Chapter 7 we meet three of he most fascinating numbers in mathematics: e, π and γ. Chapters 8 and 9 introduce two topics that normally don't feature in standard undergraduate courses - infinite products and continued fractions (respectively.) Chapter 10 begins the study of the remarkable theory of infinite numbers. Chapter 11 is perhaps, from a conceptual perspective, the most difficult chapter in the book as it deals with the rigorous construction of the real numbers using Dedekind cuts and the vital concept of completeness. Chapter 12 is a rapid survey of the further development of analysis into the realm of functions. continuity and the calculus. In Chapter 13 we give a brief account of the history of the subject and in Chapter 14 we review some of the literature that the reader might turn to next after reading this book.

You learn mathematics by doing and not by reading and so each chapter in Part 1 closes with a set of exercises which you are strongly encouraged to attempt. As well as practising techniques, these also enable you to further develop some aspects of the theory that are omitted from the main text (but explicit guidance is generally provided.) So for example, in the exercises for Chapters 4 and 5 you meet the useful concept of a subsequence and can prove the Bolzano-Weierstrass theorem for yourself. Hints and solutions to selected exercises can be found at the end of the book. Professional educators can obtain full solutions from the publishers.

I would expect that most readers will have ready access to the internet and so I have included a lot of references to Wikipedia throughout the text. This is so that you can very quickly find out a lot more about a topic if you want to. However bear in mind that Wikipedia is not yet thoroughly reliable and you should never quote mathematical results found there unless you have also confirmed them by consulting an authoritative text.

Note for Professional Educators

As pointed out above, the book falls naturally into two parts. Part 1 shadows a first year university mathematics course on sequences and series but with a little bit of number theory thrown into the mix. No calculus is used in Part 1 except at the very end in an optional aside. There is also no set theory until the very end of the book. Part II is a collection of subsidiary topics. I feel freer to use calculus here - but only occasionally. I avoid the axiomatic method throughout this book. This is a pedagogic device rather than a philosophical standpoint. I believe it is more helpful for those encountering the properties of real numbers for the first time to first develop basic analytical insight into their manipulation. The niceties of complete ordered fields can then be left to a later stage of their education. Indeed as Ivor Grattan-Guinness writes in “The Rainbow of Mathematics” (p.740), “The teaching of axioms should come *after* conveying the theory in a looser version.”

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: PART 1: Approaching Limits