MAS221 Analysis – Exercises I

Problems for Chapter 1

1. Let $a, b \in \mathbb{Q}$ be fixed, with $a < b$.

   (a) Prove that there exists $c \in \mathbb{Q}$, with $a < c < b$. [Hint: Consider the mid-point of a and b.]

   (b) Prove that for each $n \in \mathbb{N}$ with $n \geq 2$, there are $n-1$ distinct rational numbers $\{c_j, j = 1, 2, \ldots, n-1\}$ so that $a < c_j < b$.

   (c) Prove that there are infinitely many distinct rational numbers lying between a and b. [Hint: Seek a proof by contradiction.]

2. If $x > 0$ is an irrational number, show that $\sqrt{x}$ is also irrational. Hence deduce that $\sqrt{p} + \sqrt{q}$ is irrational, where p and q are distinct prime numbers. [Hint: use proof by contradiction for the first part; for the second part, try squaring.]

3. Can you find two irrational numbers a and b such that a^b is rational? [Hint: Thinking about the number $\sqrt{2}^{\sqrt{2}}$ is a good place to start.]

4. Give a careful proof of the triangle inequality:
$$|a + b| \leq |a| + |b|,$$
for all $a, b \in \mathbb{R}$, by considering all possible cases: (i) $a, b \geq 0$, (ii) $a, b < 0$, (iii) $a \geq 0, b < 0, |a| \geq |b|$, etc.

5. Prove Bernoulli's inequality:
$$(1 + x)^n \geq 1 + nx,$$
for all $n \in \mathbb{N}, x > -1$, by using mathematical induction. Give a direct proof, using the binomial theorem, for the case $x > 0$.

6. Consider the quadratic function
$$f(x) = ax^2 + bx + c,$$
where a, b and c are real numbers with $a > 0$. Show that
$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$
and hence deduce that $f(x) \geq 0$ for all x if and only if $b^2 \leq 4ac$.

7. A very famous and useful result is *Cauchy's inequality* for sums: If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers then

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

Verify that Cauchy's inequality is correct by applying the results of the previous exercise to the quadratic function $f(x) = \sum_{i=1}^{n}(a_i x + b_i)^2$.

8. (This problem is important for developing skills to help manipulate limits.)

   (a) Let $c \in \mathbb{R}$ with $c > 0$. Prove that a real number x satisfies $|x| < c$, if and only if $-c < x < c$.

   (b) Let $l \in \mathbb{R}$ and $\epsilon > 0$ (we may think of l as being a limit, and ϵ as a number that can be made arbitrarily small). Show that

$$|x - l| < \epsilon, \text{ if and only if } l - \epsilon < x < l + \epsilon.$$

9. For each of the following sets of real numbers, state whether it: (i) is finite; (ii) has a maximum element; (iii) has a minimum element. Give the maximum and minimum results when they exist.
(a) $\{2, 4, 6, 8\}$; (b) $(10, 66]$; (c) $\mathbb{Q}$; (d) $[-2000, 2000]$;
(e) $\{3, 6, 9, \ldots, 99\}$; (f) $(-\infty, 6)$; (g) $(2, 8)$;
(h) $\{0\} \cup (1, 2) \cup \{3\}$.
(i) The set of those real numbers whose square is less than or equal to -1.

10. For each of the sets E considered in the previous question decide whether it is: (i) bounded above; (ii) bounded below; (iii) bounded. Give examples of upper and lower bounds where appropriate. For those sets E which are bounded, give a real number M such that $|x| \leq M$ for all x in E.

11. For each of the following statements, concerning sets of real numbers, decide whether it is true or false. For the true ones, give a brief reason; for the false ones give a counterexample.

   (a) A set which has both a maximum element and a minimum element is necessarily finite.

   (b) If a set has a minimum element, then it has an infimum.

   (c) The infimum of a set always belongs to the set.

(d) If a set has both an infimum and a supremum, then the former is less than the latter.

(e) A non-empty set of rationals that has a rational upper bound has a rational supremum.

(f) There is a set which has 3 for an upper bound and 4 for a lower bound.

12. Show that $A \subseteq \mathbb{R}$ is bounded if and only if there exists $K > 0$ so that $|x| \leq K$ for all $x \in A$. [Hint: Use the result of Problem 8(a).]

13. Suppose that $A \subset \mathbb{R}$ is bounded above, and that $\alpha > 0$. Define the set

$$\alpha A = \{\alpha x; x \in A\}.$$

Prove that αA is bounded above and find $\sup(\alpha A)$ in terms of α and $\sup(A)$. What happens when $\alpha < 0$?

[Hint: You should first make an intelligent guess as to what $\sup(\alpha A)$ might be. Then prove that your guess is right by showing that the existence of a smaller upper bound for the set αA leads to a contradiction.]

14. If A and B are subsets of $\mathbb{R}$ that are both bounded below, show that their union $A \cup B$ is also bounded below. Prove that

$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}.$$

15. (a) Prove that the following converse statement to Proposition 1.4.2 is true: If $A \subset \mathbb{R}$ is non-empty and bounded above, and s is an upper bound for A which is such that for all $\epsilon > 0$, there exists $a \in A$ so that $a > s - \epsilon$, then $s = \sup(A)$. [Hint: Try a proof by contradiction.]

(b) Formulate and prove analogues of Proposition 1.4.2 and of (a), in the case where A is bounded below.

16. For each of the following statements, concerning sets of real numbers, decide whether it is true or false. For the true ones, give a brief reason; for the false ones, provide a counterexample.

(a) A set which does not have an infimum must be infinite.

(b) There is a set which has precisely one lower bound.

(c) A set that has an infimum and is bounded above has a supremum.

(d) It is possible for the maximum element of a set to be a lower bound for that set.

(e) There is a set E which has both a maximum and an infimum, but no minimum, which is such that $\sup E - \inf E = \pi$.

(f) If a set has *both* a maximum element and a minimum element, then the average of these two numbers must belong to the set.

17. Let E and F be two non-empty, bounded subsets of $\mathbb{R}$. Decide which of the following are true. For the true ones, give a proof; for the false ones give a counterexample.

(a) If $E \subseteq F$ then $\sup E \leq \sup F$.

(b) If $E \subset F$ then $\sup E < \sup F$.

(c) If $\sup E < \sup F$ then there exists an element y in F which is an upper bound for E.

(d) If $\sup E \leq \sup F$ then there exists an element y in F which is an upper bound for E.

(e) If $E - F := \{x - y; x \in E, y \in F\}$, then $\sup(E - F) = \sup E - \sup F$.

18. For any $a \in \mathbb{R}$ consider the set

$$A = \left\{ a - \frac{1}{n}; n \in \mathbb{N} \right\}.$$

Is A bounded above, below, or both? Can you guess the sup and inf of A (if either exist)? Prove rigorously that all of your assertions are correct.

19. Let (a_n) be a sequence of real numbers that is bounded above, i.e. there exists $M \in \mathbb{R}$ so that $a_n \leq M$ for all $n \in \mathbb{N}$. Define

$$\sup_{n\in\mathbb{N}}(a_n) = \sup\{a_n, n \in \mathbb{N}\}.$$

If (b_n) is another sequence that is bounded above, show that

$$\sup_{n\in\mathbb{N}}(a_n + b_n) \leq \sup_{n\in\mathbb{N}}(a_n) + \sup_{n\in\mathbb{N}}(b_n).$$

Give an example to show that strict inequality can occur. Work through this question again, replacing "bounded above" with "bounded below" (i.e. there exists $L \in \mathbb{R}$ so that $a_n \geq L$ for all $n \in \mathbb{N}$).

20. Consider the sequence (a_n), with general term $a_n = 1 + \frac{3}{n}$. Can you guess the limit l of this sequence?

    (a) Verify that your guess is feasible by finding a $N \in \mathbb{N}$, for each of the following given values of ϵ such that $n > N \Rightarrow |a_n - l| < \epsilon$:
    (i) $\epsilon = 0.1$ (ii) $\epsilon = 0.01$ (iii) $\epsilon = 0.001$ (iv) $\epsilon = 0.0001$ (v) $\epsilon = 10^{-10}$

    (b) Give a rigorous proof that (a_n) converges to l.

21. First guess the limits of the sequences whose nth terms are as follows

$$\text{(a)}\quad 1 - \frac{1}{n} \quad \text{(b)}\quad \frac{3}{n} \quad \text{(c)}\quad \frac{1}{n^2} \quad \text{(d)}\quad \frac{1}{\sqrt{n}}.$$

    Then, in each case, use the definition of convergence to *prove* that your guesses are correct. So given any $\epsilon > 0$, *you* need to find the N that works.

22. Write down a formula for the general term of a sequence (a_n) so that a_1, a_2, a_3, a_4 and a_5 are precisely $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}$ and use the definition of limit to prove that the sequence converges to $\frac{1}{2}$.

23. Show that if (x_n) converges to x then $(|x_n|)$ converges to $|x|$. Is the converse true? If so, give a proof and if not, present a counter-example.[Hint: For the first part, use Theorem 1.3.1.]

24. If $a_n \to 0$ as $n \to \infty$ and $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$, prove that $\lim_{n\to\infty} b_n = 0$.

    [Hint: In problems like this, its often a good idea to go back to the definition of convergence, and work with ϵ and N.]

25. (a) If $a, b \geq 0$, prove that

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}.$$

    [Hint: Square both sides.]

    (b) If $a, b \in \mathbb{R}$, deduce that

$$|\sqrt{|a|} - \sqrt{|b|}| \leq \sqrt{|a-b|}.$$

    (Hint: Imitate the proof of Theorem 1.3.1.)

(c) Prove that if the sequence (a_n) converges to l, then $(\sqrt{|a_n|})$ converges to $\sqrt{|l|}$. [Hint: Use the result of (b).]

26. Use the algebra of limits to find the limits of the following sequences:

$$\text{(a)} \quad \left(2 - \frac{1}{n}\right)\left(3 + \frac{1}{n}\right) \quad \text{(b)} \left(1 + \frac{1}{\sqrt{n}}\right)^2 \quad \text{(c)} \quad \frac{2n+3}{5n+9}$$

$$\text{(d)} \quad \frac{n^2+1}{2n^2-n+2} \quad \text{(e)} \quad \sqrt{n+1} - \sqrt{n}$$

27. The following were all written down in an examination in answer to the question "What is the definition of a sequence (x_n) converging to a limit x?" Say what is wrong, if anything, with each of them. (a) For some $\epsilon > 0$ there is an N such that $|x_n - x| < \epsilon$ for $n > N$. (b) Where $\epsilon > 0$, for some natural number N where $n > N, |x_n - x| < \epsilon$. (c) For every positive number ϵ there is a term in the sequence after which all the following terms are within ϵ of x. (d) For any $\epsilon > 0$ there is some $n > N$ such that $|x_n - x| < \epsilon$. (e) For some $\epsilon > 0$ there is a natural number $N < n$ such that $|x_n - x| < \epsilon$ for all n past a certain point.

28. The purpose of this question is to show that the order of the words in the definition of convergence is critical. A sequence (x_n) is defined to be *ridiculously-convergent* to x (this is just made up for this question) if there exists $N \in \mathbb{N}$ such that for every $\epsilon > 0$ we have $|x_n - x| < \epsilon$ whenever $n > N$.

    (a) Comment on the difference between "ridiculous convergence" and "convergence"(in the usual sense.)

    (b) Show that the sequence $(\frac{1}{n})$ is not ridiculously-convergent to 0.

29. Suppose that the sequence (x_n) converges to x. Let $C > 0$ be a fixed positive constant. Show that for any $\epsilon > 0$ there is a natural number N such that $|x_n - x| < C\epsilon$ whenever $n > N$.

    [Comment: This is really just playing with the definition of convergence, but the point is that it is sometimes useful to be able to do this, as in the next problem. The intuition behind it is, that if ϵ is thought of as a quantity that can be made as "small as you like", then so is $C\epsilon$.]

30. A sequence (a_n) is said to be *null* if it converges to zero. Prove that if (a_n) is null, and (b_n) is bounded (but not necessarily convergent), then the sequence $(a_n b_n)$ is null. Give a counter–example to show that

if both instances of “null” in the above are replaced by “convergent”, then the claim is false. [Hint: Use the definition of convergence, and Problem 29.]

31. (a) Show that if (x_n) is a sequence converging to l where each $x_n \geq 0$, then $l \geq 0$. (Hint: Try a proof by contradiction.) (b) Deduce that if (x_n) is a sequence converging to x and that $x_n < a$ for all $n \in \mathbb{N}$, then $x \leq a$. Is it true that $x < a$? For the last assertion, give a proof if it is true, or a counter–example if it is false.

32. Consider a positive sequence (x_n), i.e. one for which each $x_n > 0$, and assume that the sequence converges to a positive limit. Show that $\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n} = 1$. Give examples, one in each case, of a convergent positive sequence (x_n) for which the sequence whose nth term is $\dfrac{x_{n+1}}{x_n}$ (i) converges to zero, (ii) converges to a half, (iii) diverges (trickier.)

33. (a) Let $r > 1$ and consider the sequence $(r^{\frac{1}{n}})$. Prove that it converges to 1. [Hint: Write $r^{\frac{1}{n}} = 1 + c_n$ where $c_n > 0$ and use Bernoulli's inequality from Problem 5 to show that $\lim_{n\to\infty} c_n = 0$.]

    (b) Show that $\lim_{n\to\infty} r^{\frac{1}{n}} = 1$ when $0 < r < 1$. [Hint: Write $r = \frac{1}{s}$.]

    (c) Prove that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. [Hint: Write $\sqrt{n}^{\frac{1}{n}} = 1 + c_n$.]

34. Let $x_1 = 2.5$ and $x_{n+1} = \frac{1}{5}(x_n^2 + 6)$ for $n > 1$.

    (a) Show that each $2 \leq x_n \leq 3$. (Hint: Try a proof by contradiction.)

    (b) Show that $x_{n+1} - x_n = \frac{1}{5}(x_n - 2)(x_n - 3)$.

    (c) Show that the sequence (x_n) is monotone and find its limit as $n \to \infty$.

35. Prove that if a sequence (a_n) is monotonic decreasing, and bounded below, then it converges to its infimum. [Hint: One way of doing this is to imitate the proof of Theorem 2.3.1(1).]

36. Let $a \geq b > 0$. We define sequences (a_n) and (b_n) by taking by taking a_1 and b_1 to be a and b respectively, and requiring that for $n \geq 1$ $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = \sqrt{a_n b_n}$. In other words, a_{n+1} is the arithmetic mean of a_n and b_n while b_{n+1} is their geometric mean.

    (a) Prove that $b_n \leq b_{n+1} \leq a_{n+1} \leq a_n$ for each n.

(b) Prove that $a_{n+1} - b_{n+1} \le \frac{1}{2}(a_n - b_n)$ for all n.

(c) Deduce that the sequences (a_n) and (b_n) are each convergent and that they converge to the same limit. (The common limit $M(a;b) = \lim_{n\to} a_n = \lim_{n\to\infty} b_n$ is called the *arithmetic-geometric mean* of a and b. It can be given a precise form using objects called *elliptic integrals*.)

37. Show that if (a_n) is a sequence that is both monotonic increasing and also convergent to a limit l as $n \to \infty$, then (a_n) is bounded above and $l = \sup_{n\in\mathbb{N}}(a_n)$. What happens when (a_n) is monotonic decreasing and convergent?

38. The purpose of this question is to prove that $n^p x^n \to 0$ as $n \to \infty$ for any positive real number p and for any $-1 < x < 1$. Assume firstly that $0 < x < 1$, and write $a_n = n^p x^n$.(a) Show that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x$. (b) Deduce that $\dfrac{a_{n+1}}{a_n}$ is eventually less than one and so (a_n) is eventually decreasing. [Here 'eventually' means there is some N such that the statement is true for all $n > N$.] (c) Deduce that the sequence (a_n) tends to a non-negative limit l. (d) Use part (a) with Problem 30 to deduce that $l = 0$. What about the case where $-1 < x < 0$?

39. Suppose that (a_n) is a monotonic increasing sequence that has a subsequence (a_{n_k}) which converges to a limit l.

(a) Show that $a_n \le l$ for all $n \in \mathbb{N}$. [Hint: Use the result of Problem 37.]

(b) Show that (a_n) converges to l as $n \to \infty$. [Hint: Use the definition of convergence.]

(c) What happens when "increasing" is replaced by "decreasing" in this question?

40. Let (x_n) be a bounded sequence and define two associated sequences as follows

$$a_n = \sup\{x_m; m \ge n\} \quad \text{and} \quad b_n = \inf\{x_m; m \ge n\}$$

(a) Show that (a_n) is monotonic decreasing, bounded below and hence convergent.

(b) Show that (b_n) is monotonic increasing and bounded above and hence convergent.

We define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n,$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n.$$

Find lim sup and lim inf of the following sequences:

$$\text{(i) } (-1)^n, \text{ (ii) } \tfrac{1}{n} \text{ (iii) } (-1)^n(1-\tfrac{1}{n})$$

Note: lim sup and lim inf play a major role in some parts of advanced analysis. An important theorem states that a bounded sequence (x_n) converges to the limit l if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l.$$

You may encounter some books in which $\limsup_{n\to\infty} x_n$ is written $\overline{\lim}_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ is written $\underline{\lim}_{n\to\infty} x_n$.

41. Prove that every Cauchy sequence is bounded. [Hint: Imitate the proof of the fact that every convergent sequence is bounded.]

42. Prove that every convergent sequence is Cauchy. [Hint: Suppose (a_n) converges to a; think about how you might show that $|a_n - a_m|$ is bounded by the sum of two terms, each smaller than $\epsilon/2$, for sufficiently large m and n.]

43. Show that $(0, 1]$ is not complete by finding an example of a Cauchy sequence, all of whose terms lie in this interval, which converges to a limit that is not in the interval.