



The  
University  
Of  
Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester 2016–17

MAS331 Metric Spaces

2 hours 30 minutes

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk  
Do not remove it from the hall

Registration number from U-Card (9 digits)  
to be completed by student

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- 1 (i) The following is an incorrect definition of a metric  $d$ , defined on a set  $X$ . Identify each place where the definition is wrong, and replace this with a corrected version. (Just writing out the correct definition of a metric space, without any commentary, is not an acceptable solution.):

“A metric is a mapping  $d : X \times X \rightarrow \mathbb{R}$  such that

- (M1)  $d(x, y) > 0$  for all  $x, y \in X$  and if  $x = y$ , then  $d(x, y) = 0$ ,  
 (M2)  $d(x, y) + d(y, x) = 0$ , for all  $x, y \in X$ ,  
 (M3)  $d(x, z) - d(y, z) - d(x, y) \geq 0$ , for all  $x, y, z \in X$ .” **(5 marks)**

- (ii) Let  $(X, d)$  be a metric space. Derive the inequality

$$|d(x, z) - d(y, z)| \leq d(x, y),$$

for all  $x, y, z \in X$ . **(6 marks)**

- (iii) (a) Define  $d : C[0, 1] \times C[0, 1] \rightarrow \mathbb{R}$  where

$$d(f, g) = \int_0^1 |f(x) - g(x)|\rho(x)dx,$$

where  $\rho \in C[0, 1]$  satisfies  $\rho(x) > 0$  for all  $x \in [0, 1]$ . Prove that  $d$  is a metric. [Hint: You may use the fact that if  $h : [a, b] \rightarrow \mathbb{R}$  is a continuous function for which  $h(x) \geq 0$  for all  $a \leq x \leq b$  then  $\int_a^b h(x)dx = 0$  implies that  $h(x) = 0$  for all  $a \leq x \leq b$ .]

**(9 marks)**

- (b) For each  $r > 0$ , consider the metric  $d_r$  on  $C[0, 1]$  obtained by taking  $\rho(x) = \frac{1}{2^r}(1+x)^r$ , for each  $x \in [0, 1]$  in (iii)(a).

( $\alpha$ ) Calculate  $d_r(f, g)$  where  $f(x) = 1 + x^2$  and  $g(x) = 2 + x^2$ . **(2 marks)**

( $\beta$ ) Is  $d_\infty$  a metric on  $C[0, 1]$ , where  $d_\infty(f, g) = \lim_{r \rightarrow \infty} d_r(f, g)$ ? Give some evidence to support your conclusion. **(3 marks)**

2 Let  $(X, d)$  be a metric space.

(i) (a) Explain what it means for a subset of  $X$  to be (I) *open*, (II) *closed*.  
(4 marks)

(b) Show that if  $A \subseteq X$  is open, then its complement,  $A^c := X \setminus A$  is closed.  
(5 marks)

(c) Is the following statement correct? "Every subset of a metric space is either open or closed." Give a reason for your answer. (2 marks)

(ii) Equip  $\mathbb{R}^2$  with the usual Euclidean metric.

(a) Fix  $a, b > 0$ . Show that the set of points  $A$  is closed, where

$$A = \left\{ (x, y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}.$$

( $A$  is the set of all points that lie on an ellipse whose shape is determined by  $a$  and  $b$ .)  
(4 marks)

(b) Is the set

$$\hat{A} = \left\{ (x, y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\},$$

open, closed or neither? Give a brief explanation for your answer.  
(3 marks)

(iii) Consider the space  $C[0, 1]$  with the supremum metric

$$d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|.$$

(a) Write down an expression for the open ball  $B_r(0)$  of radius  $r > 0$ , centred on the zero function  $g(x) = 0$  for all  $x \in [0, 1]$ . (2 marks)

(b) Consider the function  $h(x) = 6x - x^2$  for all  $x \in [0, 1]$ . Is it true that  $h \in B_4(0)$ , or that  $h \in B_6(0)$ ? Give explicit calculations to support your answer. (5 marks)

- 3 (i) Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be metric spaces. Write down two (equivalent) definitions of what it means for a function  $f : X_1 \rightarrow X_2$  to be *continuous*. One of these should involve sequences, while the other uses  $\epsilon$ 's and  $\delta$ 's. (4 marks)

- (ii) Show that the following functions are continuous:

(a)  $f : (\mathbb{R}^2, d_2) \rightarrow (\mathbb{R}, d_1)$ , where for all  $x, y \in \mathbb{R}$ ,

$$f(x, y) = \cos(2x^2 - 3y).$$

Here  $d_2$  is the Euclidean metric on  $\mathbb{R}^2$ , and  $d_1$  is the standard metric on  $\mathbb{R}$ . (4 marks)

(b)  $I : (C[0, 1], d_\infty) \rightarrow (\mathbb{R}, d_1)$ , where

$$I(f) = \int_0^1 f(x) dx,$$

and  $d_\infty$  is the supremum metric, given by

$$d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|.$$

(5 marks)

- (iii) If  $(X_1, d_1)$ ,  $(X_2, d_2)$  and  $(X_3, d_3)$  are metric spaces, and  $f : X_2 \rightarrow X_3$  and  $g : X_1 \rightarrow X_2$  are continuous show that  $f \circ g$  is continuous from  $X_1$  to  $X_3$ , where for all  $x \in X_1$ ,

$$(f \circ g)(x) := f(g(x)).$$

Hence show that the mapping  $f \rightarrow \left( \int_0^1 f(x) dx \right)^2$  is continuous from  $(C[0, 1], d_\infty)$  to  $(\mathbb{R}, d_1)$ . (6 marks)

- (iv) Explain what it means for a subset of a metric space to be *compact*. If  $(X_1, d_1)$  and  $(X_2, d_2)$  are metric spaces,  $A \subseteq X_1$  is compact, and  $f : X_1 \rightarrow X_2$  is continuous, show that  $f(A)$  is compact, where

$$f(A) := \{y \in X_2; y = f(x) \text{ for some } x \in A\}.$$

(6 marks)

4 Let  $(X, d)$  be a metric space.

(i) (a) If  $T : X \rightarrow X$  is a mapping, explain what it means for  $T$ , (I) to have a *fixed point*, (II) to be a *contraction*. **(2 marks)**

(b) Prove that every contraction is continuous. **(4 marks)**

(c) State (without proof) the *contraction mapping theorem*. **(2 marks)**

(ii) Fix  $x_1 \in X$  and let  $f : X \rightarrow X$  be continuous. Define a sequence  $(x_n)$  by  $x_{n+1} = f(x_n)$  for  $n \in \mathbb{N}$ . Show that if  $(x_n)$  converges to some  $x \in X$ , then  $x$  is a fixed point of  $f$ . **(3 marks)**

(iii) Fix  $a \in \mathbb{R}$ . Show that a continuous function  $f : [a, \infty) \rightarrow \mathbb{R}$ , which is differentiable on  $(a, \infty)$ , is a contraction if and only if there exists  $0 \leq k < 1$  so that  $|f'(x)| \leq k$ , for all  $x > a$ . [Hint: Use the mean value theorem, which states that if  $a \leq b < c$ , then there exists  $y \in (b, c)$  so that  $f(c) - f(b) = f'(y)(c - b)$ .] **(6 marks)**

(iv) Which of the following are contractions from  $[a, \infty)$  to  $\mathbb{R}$ ?

(a)  $f(x) = 1 + e^{-2x}$ , with  $a = 1$ , **(2 marks)**

(b)  $f(x) = \frac{x - 3}{2x + 1}$ , with  $a = 0.25$ . **(3 marks)**

(v) Define  $k : [1, \infty) \rightarrow \mathbb{R}$  by

$$k(x) = 1 - x + e^{-2x}.$$

What can you say about the set  $k^{-1}(\{0\})$ ? **(3 marks)**

5 Let  $(X, d)$  be a metric space.

(i) Explain what it means for

(a) A sequence  $(x_n)$  in  $X$  to be a *Cauchy sequence*, (2 marks)

(b) The metric space to be *complete*. (1 mark)

(ii) Show that a closed subspace of a complete metric space is itself complete (with respect to the restricted metric). (3 marks)

(iii) Let  $a, b \in \mathbb{R}$ , with  $a < b$ . Show that  $A := \{f \in C[0, 1]; a \leq f(x) \leq b \text{ for all } x \in [1/2, 3/4]\}$  is complete with respect to the  $d_\infty$  metric. (5 marks)

(iv) Recall that the metric  $d_1$  on  $C[-1, 1]$  is given by

$$d_1(f, g) = \int_{-1}^1 |f(x) - g(x)| dx.$$

For each  $n \in \mathbb{N}$ , let  $f_n \in C[-1, 1]$  be given by

$$f_n(x) = \begin{cases} 0 & \text{for } -1 \leq x \leq 0, \\ nx & \text{for } 0 \leq x \leq \frac{1}{n}, \\ 1 & \text{for } \frac{1}{n} \leq x \leq 1. \end{cases}$$

(a) Show that  $d_1(f_n, f_m) = \frac{1}{2} \left| \frac{1}{n} - \frac{1}{m} \right|$ . (4 marks)

(b) Give a careful proof that  $(f_n)$  is a Cauchy sequence. (3 marks)

(c) Deduce that  $(C[-1, 1], d_1)$  is not a complete metric space. [Hint: You may use the fact that for  $f \in C[-1, 1]$  and  $-1 \leq a < b \leq 1$ ,

if  $\int_a^b |f(t)| dt = 0$ , then  $f(t) = 0$  for all  $a \leq t \leq b$ .] (7 marks)

**End of Question Paper**