Solutions to MAS 331/6352, Metric Spaces, 2016-17

1. (i) Change as follows:

(M1) $d(x,y) > 0$ to $d(x,y) \geq 0$ and "if $x = y$, then ..." to "$d(x,y) = 0$ if and only if $x = y$."

(M2) Change $+$ to $-$.

(M3) Change $-d(y,z)$ to $+d(y,z)$ and $\geq$ to $\leq$.

(ii) By (M3) $d(x,z) \leq d(x,y) + d(y,z)$ and so

$$d(x,z) - d(y,z) \leq d(x,y) \ldots \text{(i)}$$

Since this holds for all $x, y, z \in X$ we may interchange x and y in (i) to get

$$d(y,z) - d(x,z) \leq d(y,x) = d(x,y) \ldots \text{(ii)}$$

by (M2). Combining (i) and (ii) together gives the desired result:

$$|d(x,z) - d(y,z)| \leq d(x,y).$$

(iii) (a)(M1) $d(f,g) \geq 0$ as integrals of non-negative functions are non-negative.
Clearly $d(f,f) = 0$. If $d(f,g) = 0$ define $h(x) = |f(x) - g(x)|\rho(x)$ and use the hint together with the fact that ρ is strictly positive.

(M2) $d(f,g) = \int_a^b |f(x) - g(x)|\rho(x)dx = \int_a^b |g(x) - f(x)|\rho(x)dx = d(g,f)$.

(M3)

$$\begin{aligned} d(f,k) &= \int_a^b |f(x) - k(x)|\rho(x)dx \\ &= \int_a^b |f(x) - g(x) + g(x) - k(x)|\rho(x)dx \\ &\leq \int_a^b |f(x) - g(x)|\rho(x)dx + \int_a^b |g(x) - k(x)|\rho(x)dx \\ &= d(f,g) + d(g,k). \end{aligned}$$

(b) (α)

$$\begin{aligned} d_r(f,g) &= \frac{1}{2^r}\int_0^1 (1+x)^r dr \\ &= \frac{1}{(1+r)2^r}[(1+x)^{r+1}]_0^1 \\ &= \frac{1}{(1+r)2^r}(2^{r+1}-1) \end{aligned}$$

(β) d_∞ is not a metric as (M1) fails. Using the result of (α),

$$d_\infty(f,g) = \lim_{r\to\infty} \frac{2-2^{-r}}{1+r} = 0,$$

but $f \neq g$.

2. (i) (a) $A \subseteq X$ is (i) open, if given any $x \in A$, there exists $r > 0$ so that the open ball $B(x,r) \subseteq A$. A is (ii) closed if given any sequence (x_n) in X such that $x_n \in A$ for all $n \in \mathbb{N}$, if (x_n) converges to x, then $x \in A$.

(b) Suppose that A is open, but A^c is not closed. Let (x_n) be a sequence in A^c converging to x. Suppose that $x \in A$. Then we can find $r > 0$ such that $B(x,r) \subset A$. As (x_n) converges to x, there exists $N \in \mathbb{N}$ with $x_n \in B(x,r)$ for all $n > N$. But then $x_n \in A$, contradicting our assumption that $x_n \in A^c$. Therefore $x \notin A$ i.e. $x \in A^c$, and so A^c is closed

(c) No it's false, e.g. $\mathbb{R}$ with usual metric, $(0,1]$ is neither open nor closed.

(ii) (a) Let (x_n, y_n) be a sequence in A converging to (x,y) in $\mathbb{R}^2$ as $n \to \infty$. Then (*using a theorem from the course*) the sequence (x_n) converges to x, and (y_n) converges to y. By algebra of limits, $x_n^2/a^2 + y_n^2/b^2 \to x^2/a^2 + y^2/b^2$. Hence $(x,y) \in A$ and so A is closed.

Alternatively $f((x,y)) = x^2/a^2 + y^2/b^2$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$ by a sequential argument (as above). The set $\{1\}$ is closed in $\mathbb{R}$, and so by a theorem in the notes, $A = f^{-1}(\{1\})$ is closed.

(b) It is open. This is because it is the complement of $\left\{(x,y) \in \mathbb{R}^2; \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1\right\}$, which is closed by a similar argument to that of (a).

(iii) (a)

$$\begin{aligned} B_r(0) &= \{f \in C[0,1]; d_\infty(f,0) < r\} \\ &= \{f \in C[0,1]; \sup_{x\in[0,1]} |f(x)| < r\} \end{aligned}$$

(b) $h(0) = 0, h(1) = 5$.
There is a turning point (a maximum) at $h'(x) = 6 - 2x = 0$, i.e. at $x = 3$, which is outside $[0,1]$, so $d_\infty(h,0) = 5$. Thus $h \in B_6(0)$ but $h \notin B_4(0)$.

3. (i) f is continuous at $x \in X_1$ if given any sequence (x_n) converging to x, the sequence $(f(x_n))$ converges to $f(x)$ in X_2.

Equivalently, given any $\epsilon > 0$ there exists $\delta > 0$ so that if $d_1(x,y) < \delta$, then $d_2(f(x), f(y)) < \epsilon$.

For f to be continuous from X_1 to X_2 the above must hold for all $x \in X_1$.

(ii) (a) Suppose that $(x_n, y_n) \to (x,y)$ in $\mathbb{R}^2$ as $n \to \infty$. Then, $(x_n) \to x$ and $(y_n) \to y$. Then by algebra of limits, and continuity in $\mathbb{R}$ of the cosine function, $\cos(2x_n^2 - 3y_n) \to \cos(2x^2 - 3y)$. Hence $f(x_n, y_n) \to f(x,y)$, and so f is continuous.

(b) Let $\epsilon > 0$, and take any δ such that $0 < \delta < \frac{\epsilon}{b-a}$. If $g \in C[a,b]$ is such that $d_\infty(f,g) < \delta$, then $|f(t) - g(t)| < \delta$ for all $t \in [a,b]$, and so

$$\begin{aligned} d(I(f), I(g)) &= |I(f) - I(g)| \\ &= \left| \int_a^b f(t)\,dt - \int_a^b g(t)\,dt \right| \\ &= \left| \int_a^b (f(t) - g(t))\,dt \right| \\ &\leqslant \int_a^b |f(t) - g(t)|\,dt \leqslant \int_a^b \delta\,dt = \delta(b-a) < \epsilon. \end{aligned}$$

We take $a = 0$ and $b = 1$ in the question.

(iii) Let (x_n) be a sequence converging to x in X_1. Since g is continuous, $(g(x_n))$ converges to $g(x)$ in X_2. But f is continuous, and so $(f(g(x_n)))$ converges to $f(g(x))$ in X_3, and the result follows.

The continuity of the given map follows from taking g to be A from (ii)(b) and $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$.

(iv) A is compact if given any sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$, there exists a subsequence (x_{n_k}) which converges to a limit in A.
Let (y_n) be any sequence in $f(A)$. For each $n \in \mathbb{N}$, there is some $x_n \in A$ with $f(x_n) = y_n$. As A is compact, there is a subsequence (x_{n_r}) converging to x for some $x \in A$. As f is continuous, $y_{n_r} = f(x_{n_r})$ converges to $f(x) \in f(A)$. Thus (y_n) has a convergent subsequence, with limit in $f(A)$, and so $f(A)$ is compact.

4. (i) (a) (I) T has a fixed point if there exists $x \in X$ for which $Tx = x$; (II) T is a contraction if there exists $0 \leq k < 1$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$.

(b) Given any $\epsilon > 0$, choose $\delta = \epsilon/k$ if $k > 0$. Then if $d(x, y) < \delta$, we have $d(Tx, Ty) < k.\epsilon/k = \epsilon$, so T is continuous. If $k = 0$, then $d(Tx, Ty) = 0$ for all $x, y \in X$. Hence by (M1), T is a constant function, and so is continuous.

(c) Every contraction on a complete metric space has a (unique) fixed point.

(a) By continuity of f,

$$\begin{aligned} f(x) &= f\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} f(x_n) \\ &= \lim_{n\to\infty} x_{n+1} \\ &= x \end{aligned}$$

(iii) Suppose f is a contraction. Fix $x \in \mathbb{R}$, and let $h > 0$. We have

$$|f(x+h) - f(x)| \leqslant k|(x+h) - x| = k|h|,$$

and so $|\frac{f(x+h)-f(x)}{h}| \leqslant k$. If we let $h \to 0$, then this inequality becomes $|f'(x)| \leqslant k$ as required.
Conversely, suppose that $|f'(x)| \leqslant k$ for all $x > a$, and let $x > y \geq a$. By the mean value theorem, there exists c between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(c).$$

But $|f'(c)| \leqslant k$, so

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leqslant k,$$

and hence f is a contraction.

(iv) (a) $f'(x) = -2e^{-2x}$, so for $x > 1, |f'(x)| = 2/e^{2x} \le 2/e^2 < 1$, so f is a contraction.

(b) $f'(x) = \frac{(2x+1)-2(x-3)}{(2x+1)^2} = \frac{7}{(2x+1)^2}$. When e.g. $x = 1/2, f'(x) = 7/4 > 1$, so f is not a contraction.

(v)

$$\begin{aligned} k^{-1}(\{0\}) &= \{x \ge 1; k(x) = 0\} \\ &= \{x \ge 1; 1 + e^{-2x} = x\}, \end{aligned}$$

By the contraction mapping theorem and (iv)(a), we conclude that $k^{-1}(\{0\}) \neq \emptyset$. In fact by uniqueness of fixed points, it is a singleton set.

5. (i) (a) (x_n) is a Cauchy sequence if given any $\epsilon > 0$, there exists $N \in \mathbb{N}$, so that if $m, n > N$, then $d(x_m, x_n) < \epsilon$.

(b) (X, d) is complete if every Cauchy sequence in X converges to a limit in X.

(ii) Let A be a closed subspace of X and (x_n) be a Cauchy sequence in X with $x_n \in A$ for all $n \in \mathbb{N}$. As X is complete, (x_n) converges to x (say). As A is closed, $x \in A$. Hence A is complete.

(iii) We use the fact that $(C[0,1], d_\infty)$ is complete and seek to show that A is closed. Then the result follows by (ii). Let (f_n) be a sequence in $C[0,1]$ so that $f_n \in A$ for each $n \in \mathbb{N}$. Then $a \le f_n(x) \le b$ for all $n \in \mathbb{N}$ and all $x \in [1/2, 3/4]$. Assume that (f_n) converges to some f in $(C[0,1], d_\infty)$. Then (f_n) also converges to f pointwise, i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$. By elementary properties of limits, we have $a \le f(x) \le b$ for all $x \in [1/2, 3/4]$. Then $f \in A$, and so A is closed.

(iv) (a) For $m < n$,

$$\begin{aligned} d_1(f_n, f_m) &= \int_0^{\frac{1}{n}} (n-m)x\,dx + \int_{\frac{1}{n}}^{\frac{1}{m}} (1-mx)\,dx \\ &= (n-m)\left[\frac{x^2}{2}\right]_0^{\frac{1}{n}} + \left[x - \frac{mx^2}{2}\right]_{\frac{1}{n}}^{\frac{1}{m}} \\ &= (n-m)\frac{1}{2n^2} + \frac{1}{m} - \frac{1}{2m} - \frac{1}{n} + \frac{m}{2n^2} \\ &= \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right) \end{aligned}$$

and the result follows.

(b) Given $\epsilon > 0$, choose N to be the smallest integer that exceeds $\frac{1}{2\epsilon}$. If $m, n > N$ then $d(f_n, f_m) \leq \frac{1}{2n} + \frac{1}{2m} < \epsilon$ by (a) and so (f_n) is Cauchy.[1]

(c) Assume $f_n \to f$ in $C[-1,1]$ then $d_1(f_n, f) \to 0$ as $n \to \infty$ in $C[-1,1]$.
Now

$$\begin{aligned} d_1(f_n, f) &\geq \int_{-1}^{0} |f_n(t) - f(t)| dt \\ &= \int_{-1}^{0} |f(t)| dt \to 0 \text{ as } n \to \infty, \end{aligned}$$

$\Rightarrow f(t) = 0$ for $-1 \leq t \leq 0$.
Also

$$\begin{aligned} d_1(f_n, f) &\geq \int_{\frac{1}{n}}^{1} |f_n(t) - f(t)| dt \\ &= \int_{\frac{1}{n}}^{1} |1 - f(t)| dt \\ &\to \int_{0}^{1} |1 - f(t)| dt \to 0 \text{ as } n \to \infty, \end{aligned}$$

$\Rightarrow f(t) = 1$ for $0 < t \leq 1$.
Such an f would not be continuous and so we conclude that $(C[-1,1], d_1)$ is not complete.

[1] The existence of N is guaranteed by the Archimedean property of $\mathbb{R}$, but you don't need to say this to get full marks.