The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

MAS350 Measure and Probability **2 hours**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (i) Let S be a given set and consider the following statements:

(I) A σ–algebra is a collection Σ of subsets of S for which $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, whenever $A_n \in S$ for all $n \in \mathbb{N}$.

(II) A measure is a mapping $m : S \to [0, \infty)$ for which

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n),$$

whenever (A_n) is a sequence of subsets of S.

These statements are both wrong. Explain carefully how they can be corrected. ***(7 marks)***

(ii) Let Σ_1 and Σ_2 be σ–algebras of a set S.

(a) Define

$$\Sigma_1 \cap \Sigma_2 = \{A \subseteq S, A \in \Sigma_1 \text{ and } A \in \Sigma_2\}.$$

Show that $\Sigma_1 \cap \Sigma_2$ is a σ–algebra. ***(3 marks)***

(b) Define

$$\Sigma_1 \cup \Sigma_2 = \{A \subseteq S, A \in \Sigma_1 \text{ or } A \in \Sigma_2\},$$

(where "or" is inclusive). Either show that $\Sigma_1 \cup \Sigma_2$ is a σ–algebra, or give a counter–example to demonstrate that, in general, it isn't. ***(3 marks)***

(iii) Define the *symmetric difference* $A \triangle B$ between subsets A and B of S by

$$A \triangle B = (A \cup B) - (A \cap B).$$

(a) Show that $A \triangle B = (A - B) \cup (B - A)$. ***(5 marks)***

(b) Calculate the Lebesgue measure of $A \triangle B$ when $A = (0, 1)$ and $B = (-1/3, 1/2) \cup (3/4, 2)$. ***(4 marks)***

(c) If A and B are as in (b), and $S = (-1, 2)$ equipped with the uniform probability measure on its Borel σ-algebra, what is the probability of the set $A \triangle B$? ***(3 marks)***

2 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) Recall that $f : S \to \mathbb{R}$ is a measurable function if $f^{-1}((a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$. Show that this is equivalent to requiring $f^{-1}([a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$. ***(4 marks)***

(ii) Suppose that $f : S \to \mathbb{R}$ is measurable and $a \in \mathbb{R}$. Show that $f^{-1}(\{a\}) \in \Sigma$. ***(2 marks)***

(iii) Assume that both $g : S \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are measurable. Show that $f \circ g$ is measurable from S to $\mathbb{R}$. ***(3 marks)***

(iv) Let $y \in \mathbb{R}$ be fixed. If $f : \mathbb{R} \to \mathbb{R}$ is measurable, show that $h : \mathbb{R} \to \mathbb{R}$ is measurable, where $h(x) = f(x + y)$ for all $x \in \mathbb{R}$. ***(2 marks)***

(v) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, show that f and its derivative f' are both measurable. ***(4 marks)***

(vi) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is n-times differentiable, and let $f^{(r)}$ denote its rth derivative for $r = 1, 2, \ldots, n$. Deduce that $f, f^{(1)}, \ldots, f^{(n)}$ are all measurable. ***(3 marks)***

(vii) Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ and define

$$A = \{x \in S; \lim_{n \to \infty} f_n(x) \text{ exists}\}.$$

Show that $A \in \Sigma$.

[Hint: Make use of the measurable functions $\liminf_{n \to \infty} f_n$ and $\limsup_{n \to \infty} f_n$.] ***(7 marks)***

3 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra. Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is denoted by λ.

(i) Let $f : S \to \mathbb{R}$ be measurable. Explain how the Lebesgue integral of f is constructed in each of the following cases, carefully stating any restrictions on f that are needed (if necessary), and giving the range of values that the integral may take:

(a) f is a non–negative simple function, ***(3 marks)***

(b) f is a non–negative measurable function, ***(2 marks)***

(c) f is a general measurable function. ***(3 marks)***

What does it mean for f to be *integrable*? ***(1 mark)***

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$f = \begin{cases} 0 & \text{if } x < -2 \\ -7, & \text{if } -2 \le x < -1, \\ 4 & \text{if } -1 \le x < 0, \\ 11 & \text{if } 0 \le x < 1, \\ -3 & \text{if } 1 \le x < 2, \\ -2 & \text{if } 2 \le x < 5, \\ 0, & \text{if } x \ge 5 \end{cases}$$

(a) Write down the positive (f_+) and negative (f_-) parts of f in the form of simple functions. ***(2 marks)***

(b) Calculate $\displaystyle\int_S |f| d\lambda$ by using the result of (a). ***(3 marks)***

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is integrable and $g(x) = \dfrac{xf(x)}{1+x^2}$ for all $x \in \mathbb{R}$, explain why g is integrable. ***(5 marks)***

(iv) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $0 < g(x)\mathbf{1}_{[n-1,n)}(x) \le n$ for all $n = 2, 3, 4, \ldots$. Show that $\displaystyle\int_1^\infty \frac{1}{g(x)} dx = \infty$. ***(6 marks)***

[Hint: For all $n = 2, 3, 4, \ldots$, write $\displaystyle\int_1^n \frac{1}{g(x)} dx = \int_1^n \sum_{k=2}^n \frac{1}{g(x)} \mathbf{1}_{[k-1,k)}(x) dx$ and use the fact that $\displaystyle\sum_{k=2}^\infty \frac{1}{k} = \infty$.]

4 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra.

(i) State *Lebesgue's dominated convergence theorem.* ***(3 marks)***

(ii) Assume that $f : \mathbb{R} \to \mathbb{R}$ is integrable. Deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n}{e^{x^2} + n} f(x)dx = \int_{\mathbb{R}} f(x)dx.$$

What can you say about $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{e^{x^2}}{e^{x^2} + n} f(x)dx$? ***(5 marks)***

(iii) (a) Let $f : [a, b] \times S \to \mathbb{R}$ be a measurable function for which

(I) The mapping $x \to f(t, x)$ is integrable for all $t \in [a, b]$,

(II) The mapping $t \to f(t, x)$ is differentiable for all $x \in S$,

(III) There exists a non-negative integrable function $h : S \to \mathbb{R}$ so that $\left|\frac{\partial f(t, x)}{\partial t}\right| \leq h(x)$ for all $t \in [a, b], x \in S$.

Show that the mapping $t \to \int_S f(t, x)dm(x)$ is differentiable on (a, b) and that

$$\frac{d}{dt} \int_S f(t, x)dm(x) = \int_S \frac{\partial f(t, x)}{\partial t} dm(x).$$

(6 marks)

(b) Let $f : [1, \infty) \to \mathbb{R}$ be an integrable function such that $x \to \frac{f(x)}{x^2}$ is also integrable. Deduce that the mapping $t \to \int_{[1,\infty)} \sin(2tx^2) \frac{f(x)}{x^2} dx$ is differentiable, and find an expression for its derivative. ***(6 marks)***

(iv) A sequence (f_n) of integrable functions from $[0, \infty)$ to $\mathbb{R}$ is said to *converge in the $\mathcal{L}_1$ sense* to an integrable function f if

$$\lim_{n\to\infty} \int_{[0,\infty)} |f - f_n| d\lambda = 0,$$

where λ is the restriction of Lebesgue measure to $[0, \infty)$. Show that, for any integrable function f, the sequence $(f \mathbf{1}_{[0,n]})$ converges in the $\mathcal{L}_1$ sense to f. ***(5 marks)***

5 Throughout this question, $(\Omega, \mathcal{F}, P)$ is a probability space.

(i) Let (A_n) be a sequence of subsets of Ω in $\mathcal{F}$.

(a) Define the sets $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$, and briefly explain why each set is in $\mathcal{F}$. ***(3 marks)***

(b) Give a direct proof that $\limsup_{n\to\infty} P(A_n) \leq P\left(\limsup_{n\to\infty} A_n\right)$. ***(5 marks)***

(c) If $B \in \mathcal{F}$, show that

$$B - \liminf_{n\to\infty} A_n = \limsup_{n\to\infty}(B - A_n).$$

Hence deduce that $\left(\liminf_{n\to\infty} A_n\right)^c = \limsup_{n\to\infty} A_n^c$. ***(7 marks)***

(ii) If $X : \Omega \to \mathbb{R}$ is an integrable random variable and $a \in \mathbb{R}$, show that

$$\mathbb{E}(\min\{X, a\}) \leq \min\{\mathbb{E}(X), a\}.$$

(6 marks)

Use this inequality to find an upper bound for $\mathbb{E}(\min\{X, a\})$ when

(a) $a = 1$, and X is a Bernoulli random variable taking values 1 with probability 3/4 and 0 with probability 1/4, ***(1 mark)***

(b) $a = 54$, and $X = Y_1 + Y_2 + \cdots + Y_{10}$ with $Y_k \sim N(k, 1)$ for $k = 1, 2, \ldots, 10$. ***(3 marks)***

End of Question Paper