Solutions to MAS451 Exam 2015-16

1. (i) For (I) we need $A_n \in \Sigma$ for all $n \in \mathbb{N}$. But we also need $S \in \Sigma$ and $A^c \in \Sigma$ whenever $A \in \Sigma$.

For (II) we need to replace $m : S \to [0, \infty)$ with $m : \Sigma \to [0, \infty]$ and $m(\emptyset) = 0$. We also need that the sequence (A_n) is such that $A_n \in \Sigma$ for all $n \in \mathbb{N}$, and that the sets in (A_n) are mutually disjoint.

(ii) (a) Since $S \in \Sigma_1$ and $S \in \Sigma_2$, $S \in \Sigma_1 \cap \Sigma_2$. Suppose (A_n) is a sequence of sets in $\Sigma_1 \cap \Sigma_2$. Then $A_n \in \Sigma_1$ for all $n \in \mathbb{N}$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1$. But also $A_n \in \Sigma_2$ for all $n \in \mathbb{N}$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_2$. Hence $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1 \cap \Sigma_2$. If $A \in \Sigma_1 \cap \Sigma_2$, $A^c \in \Sigma_1$ and $A^c \in \Sigma_2$. Hence $A^c \in \Sigma_1 \cap \Sigma_2$.

(b) $\Sigma_1 \cup \Sigma_2$ is not in general a σ-algebra for if $A \in \Sigma_1$ and $B \in \Sigma_2$ there is no good reason why $A \cup B \in \Sigma_1 \cup \Sigma_2$. For example let $S = \{1, 2, 3\}, \Sigma_1 = \{\emptyset, \{1\}, \{2, 3\}, S\}, \Sigma_2 = \{\emptyset, \{2\}, \{1, 3\}, S\}, A = \{1\}, B = \{2\}$. Then $A \cup B = \{1, 2\}$ is neither in Σ_1 nor Σ_2.

(iii) $S \in \Sigma$ if S is finite or countable. If not $S^c = \emptyset$ is finite as it has zero elements. If $A \in \Sigma$ then either A or A^c is finite or countable, hence $A^c \in \Sigma$ by symmetry. If (A_n) is a sequence of sets in Σ then either they are all finite or countable, in which case their union is countable, or at least one set A_m is not finite or countable, in which case A_m^c is. But then

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subseteq A_m^c,$$

is finite or countable, and so is in Σ.

(iv) (a)

$$\begin{aligned} A \triangle B &= (A \cup B) - (A \cap B) \\ &= (A \cup B) \cap (A \cap B)^c \\ &= (A \cup B) \cap (A^c \cup B^c) \\ &= [(A \cup B) \cap A^c] \cup [(A \cup B) \cap B^c] \\ &= (A \cap A^c) \cup (B \cap A^c) \cup (A \cap B^c) \cup (B \cap B^c) \\ &= (B \cap A^c) \cup (A \cap B^c) = (B - A) \cup (A - B). \end{aligned}$$

(b) $B - A = (-1/3, 0] \cup [1, 2)$ and $A - B = [1/2, 3/4]$. The three intervals are mutually disjoint, and so $\lambda(A \triangle B) = 1/3 + 1 + 1/4 = 19/12$.

(c) $\lambda(S) = 3$ and $P(A \triangle B) = \lambda(A \triangle B)/3 = 19/36$.

2. (i) (a) If $f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$ where $n \in \mathbb{N}, c_i \geq 0$ and $A_i \in \Sigma$, for all $i = 1, \ldots, n$ being mutually disjoint with $\bigcup_{i=1}^{n} A_i = S$, then $\int_S f dm = \sum_{i=1}^{n} c_i m(A_i) \in [0, \infty]$.

(b) $\int_S f dm = \sup \left\{ \int_S g dm; g \text{ simple } 0 \leq g \leq f \right\} \in [0, \infty]$.

(c) Write $f = f_+ - f_-$, where $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Then $\int_S f dm = \int_S f_+ dm - \int_S f_- dm \in [-\infty, \infty]$, with the restriction that both integrals on the right hand side are not infinite.

f is integrable if $\int_S f dm \in (-\infty, \infty)$ (equivalently $\int_S |f| dm < \infty$).

(ii) $|f| = 7\mathbf{1}_{[-2,1)} + 4\mathbf{1}_{[-1,0)} + 11\mathbf{1}_{[0,1)} + 3\mathbf{1}_{[1,2)} + 2\mathbf{1}_{[2,5)}$,

$$\int_{\mathbb{R}} |f| d\lambda = 7 + 4 + 11 + 3 + [2 \times 3)] = 31.$$

(iii) g is measurable as it is a product of measurable functions. We have $\left|\frac{x}{1+x^2}\right| = \frac{|x|}{1+x^2} \leq \frac{1}{2}$, for since $(1 - |x|)^2 \geq 0$, we have $2|x| \leq 1 + x^2$. But then we have $|g(x)| \leq \frac{1}{2}|f(x)|$ for all $x \in \mathbb{R}$, and so g is integrable as by monotonicity,

$$\int_{\mathbb{R}} |g| d\lambda \leq \frac{1}{2} \int_{\mathbb{R}} |f| d\lambda < \infty.$$

(iv) (a) Let $G = 1/g$. For any $a \in \mathbb{R} - \{0\}, G(x) > a$ iff $1/g(x) > a$ iff $g(x) < 1/a$. *(in fact $a > 0$ is enough here).* It follows that $G^{-1}((a, \infty)) = g^{-1}(-\infty, 1/a) \in \Sigma$ as g is measurable. For the case $a = 0$, as g is positive so is G, hence $G^{-1}((0, \infty)) = S \in \Sigma$. Hence G is measurable. By a theorem in the course, products of measurable functions are measurable, and the result follows by (a).

(v) We use the fact that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable functions, and that $\lim_{n\to\infty} f_n(x)$ exists for some $x \in S$ if and only if $\limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$, in which case $\lim_{n\to\infty} f_n(x)$ is their common value. So

$$A = \{x \in S; \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)\} = g^{-1}(\{0\}),$$

where $g = \limsup_{n\to\infty} f_n - \liminf_{n\to\infty} f_n$ is measurable. The result follows by (ii).

3. (i) (a) Fix $E \in \Sigma_1 \otimes \Sigma_2$. Define the x and y-slices of E by

$$E_x = \{y \in S_2, (x, y) \in E\} \text{ and } E_y = \{x \in S_1, (x, y) \in E\}.$$

Then $E_x \in \Sigma_2, E_y \in \Sigma_1$. Define $\phi_E : S_1 \to \mathbb{R}$ and $\psi_E : S_2 \to \mathbb{R}$ by

$$\phi_E(x) = m_2(E_x), \psi_E(y) = m_1(E_y).$$

Both of these functions are measurable (by Dynkin's $\pi - \lambda$ theorem), but

$$\phi_E(x) = \int_{S_2} \mathbf{1}_E(x, y)dm_2(y), \quad \psi_E(y) = \int_{S_1} \mathbf{1}_E(x, y)dm_2(x),$$

and this proves the first part of the theorem.
The second comes from the fact that product measure

$$(m_1 \times m_2)(E) = \int_{S_1} \phi_E(x)dm_1(x) = \int_{S_2} \psi_E(y)dm_2(y),$$

and $(m_1 \times m_2)(E) = \int_{S_1 \times S_2} \mathbf{1}_E d(m_1 \times m_2)$.

(b) The result follows easily for simple functions by linearity. By a theorem in the course, there exists an increasing sequence of simple functions $(s_n, n \in \mathbb{N})$ converging pointwise to f. By monotone convergence $\int_{S_2} f(x, y)dm_2(y) = \lim_{n\to\infty} \int_{S_2} s_n(x, y)dm_2(y)$ and it is measurable (as a function of x) since it is the pointwise limit of a sequence of measurable functions. The same procedure works for the other mapping.
Again by the monotone convergence theorem

$$\lim_{n\to\infty} \int_{S_1 \times S_2} s_n d(m_1 \times m_2) = \int_{S_1 \times S_2} f d(m_1 \times m_2),$$

and the result follows by applying monotone convergence to each of the integrals in the array in the question (where f is replaced by s_n), using the fact that, by monotonicity, for each $i = 1, 2$, $\int_{S_i} s_n dm_i$ is increasing.

(c) Write $f = f_+ - f_-$, where $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. The theorem of (a) holds separately for each of f_+ and f_-. Then to prove the first part, by definition,

$$\int_{S_2} f(x, y)dm_1(y) = \int_{S_2} f_+(x, y)dm_1(y) - \int_{S_2} f_-(x, y)dm_1(y),$$

is measurable, as it is the difference of two measurable functions. The same argument works for the other mapping, and to prove the second part of the theorem.

(ii) (a) Since the function is non-negative and measurable, by Fubini's theorem (i) (a):

$$\int_{[0,1]\times[0,1]} e^{-x-y}\frac{xy^2}{(1+x^2)(1+y^2)} = \left(\int_0^1 e^{-x}\frac{x}{1+x^2}dx\right)\left(\int_0^1 e^{-y}\frac{y^2}{1+y^2}dy\right).$$

Each of $x \to e^{-x}\frac{x}{1+x^2}$ and $y \to e^{-y}\frac{y^2}{1+y^2}$ is continuous, hence Riemann integrable, hence Lebesgue integrable on $[0,1]$, and the result follows.

(b)

$$e^{-\left(\frac{x+y}{n}\right)}\frac{nxy^2}{(1+x^2)(n+y^2)} = e^{-\left(\frac{x+y}{n}\right)}\frac{x}{1+x^2}\frac{y^2}{1+y^2/n} \leq 1,$$

which is integrable on $[0,1]\times[0,1]$, so by dominated convergence

$$\lim_{n\to\infty}\int_0^1\int_0^1 e^{-\left(\frac{x+y}{n}\right)}\frac{nxy^2}{(1+x^2)(n+y^2)}dxdy = \int_0^1 \frac{x}{1+x^2}dx\int_0^1 y^2dy = \frac{1}{6}\log(2).$$

4. (i) (a) $\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}}\bigcup_{k=n}^{\infty} A_k$, $\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}}\bigcap_{k=n}^{\infty} A_k$. Both sets are in $\mathcal{F}$ as the σ–algebra is closed under finite and countable unions and intersections.

(b) As the events $\bigcup_{k=n}^{\infty} A_k$ form a decreasing sequence, by continuity of probability we have

$$\begin{aligned} P\left(\limsup_{n\to\infty} A_n\right) &= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &= \limsup_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \\ &\geq \limsup_{n\to\infty} P(A_n), \end{aligned}$$

where the last line is by monotonicity.

(c)

$$\begin{aligned} B - \liminf_{n\to\infty} A_n &= B \cap \left(\bigcup_{n\in\mathbb{N}}\bigcap_{k=n}^{\infty} A_k\right)^c \\ &= B \cap \bigcap_{n\in\mathbb{N}}\bigcup_{k=n}^{\infty} A_k^c \end{aligned}$$

$$
\begin{aligned}
&= \bigcap_{n \in \mathbb{N}} \bigcup_{k=n}^{\infty} (B \cap A_k^c) \\
&= \limsup_{n \to \infty} (B - A_n)
\end{aligned}
$$

For the last part, take $B = \Omega$.

(ii) Write $\mathbb{E}(\min\{X, a\}) = \int_\Omega \min\{X(\omega), a\} dP(\omega)$
Now for each $\omega \in \Omega, \min\{X(\omega), a\} \leq a$ and $\min\{X(\omega), a\} \leq X(\omega)$, so by monotonicity, $\mathbb{E}(\min\{X, a\}) \leq \int_\Omega a dP(\omega) = 1$ and $\mathbb{E}(\min\{X, a\}) \leq \int_\Omega X(\omega) dP(\omega) = \mathbb{E}(X)$. Hence $\mathbb{E}(\min\{X, a\}) \leq \min\{E(X), a\})$.

(a) $\mathbb{E}(X) = 3/4$ so $\mathbb{E}(\min\{X, a\}) \leq 3/4$.

(b) $\mathbb{E}(X) = \sum_{i=1}^{10} \mathbb{E}(Y_i) = 1 + 2 + \cdots + 10 = \frac{1}{2}.10.11 = 55$. Hence $\mathbb{E}(\min\{X, a\}) \leq 54$.

(iii) Let (X_n) be a sequence of random variables defined on Ω which converges pointwise to a random variable X. Suppose there is an integrable random variable Y defined on Ω, so that $|X_n| \leq Y$ for all $n \in \mathbb{N}$. Then X is integrable and

$$\mathbb{E}(X) = \lim_{n \to \infty} \mathbb{E}(X_n).$$

(*You can also assume that X_n is integrable for all $n \in \mathbb{N}$, but that is not strictly necessary as it is established within the proof of the theorem.*)

(iv) (a) Take any $\epsilon > 0$, fix $\omega \in \Omega$, then there exists $n \in \mathbb{N}$ such that $\omega \in A_m$ for all $m \geq n$. Hence $|\mathbf{1}_{A_m}(\omega) - 1| = 0 < \epsilon$. Alternatively the sequence $(\mathbf{1}_{A_n}, n \in \mathbb{N})$ is increasing and bounded above and so converges to its least upper bound, which is clearly 1.

(b) For each $\omega \in \Omega, |\mathbf{1}_{A_n}(\omega) - 1)X(\omega)| \leq 2|X(\omega)|$ and $2X$ is integrable, hence the result follows by dominated convergence since

$$\lim_{n \to \infty} \mathbb{E}(|X_n - X|) = \mathbb{E}\left(\lim_{n \to \infty} |\mathbf{1}_{A_n}(\omega) - 1|.|X|\right) = 0.$$