The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

MAS451 Measure and Probability **2 hours**

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

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1 (i) Let S be a given set and consider the following statements:

(I) A σ–algebra is a collection Σ of subsets of S for which $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, whenever $A_n \in S$ for all $n \in \mathbb{N}$.

(II) A measure is a mapping $m : S \to [0, \infty)$ for which

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n),$$

whenever (A_n) is a sequence of subsets of S.

These statements are both wrong. Explain carefully how they can be corrected. ***(7 marks)***

(ii) Let Σ_1 and Σ_2 be σ–algebras of a set S.

(a) Define

$$\Sigma_1 \cap \Sigma_2 = \{A \subseteq S, A \in \Sigma_1 \text{ and } A \in \Sigma_2\}.$$

Show that $\Sigma_1 \cap \Sigma_2$ is a σ–algebra. ***(3 marks)***

(b) Define

$$\Sigma_1 \cup \Sigma_2 = \{A \subseteq S, A \in \Sigma_1 \text{ or } A \in \Sigma_2\},$$

(where "or" is inclusive). Either show that $\Sigma_1 \cup \Sigma_2$ is a σ–algebra, or give a counter–example to demonstrate that, in general, it isn't. ***(3 marks)***

(iii) Recall that a set is *countable* if it can be put into one–to–one correspondence with the natural numbers. Let S be a set and Σ be a collection of subsets of S that is chosen as follows: $A \in \Sigma$ if either A is finite or countable or A^c is finite or countable. Show that Σ is a σ–algebra. You may use the facts that a countable union of finite or countable sets is itself finite or countable, and that a subset of a countable set is finite or countable. ***(8 marks)***

(iv) Define the *symmetric difference* $A \triangle B$ between subsets A and B of S by

$$A \triangle B = (A \cup B) - (A \cap B).$$

(a) Show that $A \triangle B = (A - B) \cup (B - A)$. ***(5 marks)***

(b) Calculate the Lebesgue measure of $A \triangle B$ when $A = (0, 1)$ and $B = (-1/3, 1/2) \cup (3/4, 2)$. ***(4 marks)***

(c) If A and B are as in (b), and $S = (-1, 2)$ equipped with the uniform probability measure on its Borel σ-algebra, what is the probability of the set $A \triangle B$? ***(3 marks)***

2 Throughout this question (S, Σ, m) is a measure space and $\mathbb{R}$ is equipped with its usual Borel σ-algebra. Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is denoted by λ.

(i) Let $f : S \to \mathbb{R}$ be measurable. Explain how the Lebesgue integral of f is constructed in each of the following cases, carefully stating any restrictions on f that are needed (if necessary), and giving the range of values that the integral may take:

(a) f is a non–negative simple function, ***(3 marks)***

(b) f is a non–negative measurable function, ***(2 marks)***

(c) f is a general measurable function. ***(3 marks)***

What does it mean for f to be *integrable*? ***(1 mark)***

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$f = \begin{cases} 0 & \text{if } x < -2 \\ -7, & \text{if } -2 \le x < -1, \\ 4 & \text{if } -1 \le x < 0, \\ 11 & \text{if } 0 \le x < 1, \\ -3 & \text{if } 1 \le x < 2, \\ -2 & \text{if } 2 \le x < 5, \\ 0, & \text{if } x \ge 5 \end{cases}$$

Write $|f|$ as a simple function and hence calculate $\int_S |f| d\lambda$. ***(3 marks)***

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g(x) = \dfrac{xf(x)}{1+x^2}$ for all $x \in \mathbb{R}$, explain why g is integrable. ***(5 marks)***

(iv) Suppose that $g : S \to (0, \infty)$ is a measurable function.

(a) Prove that $1/g$ is measurable, where $(1/g)(x) = 1/g(x)$, for all $x \in S$. ***(6 marks)***

(b) If $f : S \to \mathbb{R}$ is measurable, show that $h = f/g$ is measurable. ***(3 marks)***

(v) Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ and define

$$A = \{x \in S; \lim_{n\to\infty} f_n(x) \text{ exists}\}.$$

Show that $A \in \Sigma$. [Hint: Make use of the measurable functions $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$.] ***(7 marks)***

3 (i) Let (S_1, Σ_1, m_1) and (S_2, Σ_2, m_2) be measure spaces. A version of *Fubini's theorem* is as follows: Let $f : S_1 \times S_2 \to \mathbb{R}$ be a non-negative measurable function. Then the mappings

$$x \to \int_{S_2} f(x,y)dm_2(y) \quad \text{and} \quad y \to \int_{S_1} f(x,y)dm_1(x),$$

are both measurable. Furthermore

$$\begin{aligned} \int_{S_1 \times S_2} f d(m_1 \times m_2) &= \int_{S_1} \left(\int_{S_2} f(x,y)dm_2(y) \right) dm_1(x) \\ &= \int_{S_2} \left(\int_{S_1} f(x,y)dm_1(x) \right) dm_2(y). \end{aligned}$$

(a) Explain briefly why this theorem is true in the case where f is an indicator function. You should not give a detailed proof, but your explanation should include definitions of key concepts such as x and y-slices of a set (where $x \in S_1$ and $y \in S_2$), and product measure. ***(9 marks)***

(b) Using the result of (a), give a detailed proof of the theorem for general non-negative measurable functions f. ***(8 marks)***

(c) Prove Fubini's theorem for a real-valued, integrable function $f : S_1 \times S_2 \to \mathbb{R}$. ***(6 marks)***

(ii) (a) Explain why $(x,y) \to e^{-x-y}\dfrac{xy^2}{(1+x^2)(1+y^2)}$ is integrable (with respect to Lebesgue measure) on $(0,1) \times (0,1)$. Do not attempt to evaluate the integral. ***(4 marks)***

(b) Evaluate $\displaystyle\lim_{n\to\infty} \int_0^1 \int_0^1 e^{-\left(\frac{x+y}{n}\right)} \frac{nxy^2}{(1+x^2)(n+y^2)} dxdy.$ ***(6 marks)***

4 Throughout this question, $(\Omega, \mathcal{F}, P)$ is a probability space.

(i) Let (A_n) be a sequence of subsets of Ω in $\mathcal{F}$.

(a) Define the sets $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$, and briefly explain why each set is in $\mathcal{F}$. ***(3 marks)***

(b) Give a direct proof that $\limsup_{n\to\infty} P(A_n) \leq P\left(\limsup_{n\to\infty} A_n\right)$. ***(5 marks)***

(c) If $B \in \mathcal{F}$, show that

$$B - \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} (B - A_n).$$

Hence deduce that $\left(\liminf_{n\to\infty} A_n\right)^c = \limsup_{n\to\infty} A_n^c$. ***(7 marks)***

(ii) If $X : \Omega \to \mathbb{R}$ is an integrable random variable and $a \in \mathbb{R}$, show that

$$\mathbb{E}(\min\{X, a\}) \leq \min\{\mathbb{E}(X), a\}.$$

(6 marks)

Use this inequality to find an upper bound for $\mathbb{E}(\min\{X, a\})$ when

(a) $a = 1$, and X is a Bernoulli random variable taking values 1 with probability 3/4 and 0 with probability 1/4, ***(1 mark)***

(b) $a = 54$, and $X = Y_1 + Y_2 + \cdots + Y_{10}$ with $Y_k \sim N(k, 1)$ for $k = 1, 2, \ldots, 10$. ***(3 marks)***

(iii) State *Lebesgue's dominated convergence theorem* in a probabilistic context. ***(3 marks)***

(iv) A sequence (X_n) of random variables is said to *converge in the mean* to a random variable X if $\lim_{n\to\infty} \mathbb{E}(|X_n - X|) = 0$. Let $(A_n, n \in \mathbb{N})$ be a sequence of subsets of Ω for which $A_n \in \mathcal{F}$, $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} A_n = \Omega$.

(a) Show that $(\mathbf{1}_{A_n}, n \in \mathbb{N})$ converges pointwise to 1 on Ω. ***(2 marks)***

(b) Let X be an integrable random variable and define $X_n = X\mathbf{1}_{A_n}$ for each $n \in \mathbb{N}$. Prove that $(X_n, n \in \mathbb{N})$ converges in mean to X. ***(3 marks)***

End of Question Paper