Corrigendum to "Probabilistic Trace and Poisson Summation Formulae on Locally Compact Abelian Groups", by David Applebaum [1] "

In [1] we studied convolution semigroups of probability measures $(\mu_t, t \geq 0)$ on a locally compact abelian group G, which is equipped with a discrete subgroup Γ. Then $(\widetilde{\mu_t}, t \geq 0)$ is a convolution semigroup on the factor group G/Γ, wherein for each $t \geq 0, \widetilde{\mu_t} := \mu_t \circ \pi^{-1}$, where $\pi : G \to G/\Gamma$ is the natural surjection. If μ_t has a continuous density f_t for all $t > 0$, then its Γ–periodisation F_t, which is the density of $\widetilde{\mu_t}$ on G/Γ, satisfies the probabilistic trace formula

$$F_t = \mathrm{trace}(P_t),$$

where $(P_t, t \geq 0)$ is the contraction semigroup on $L^2(G/\Gamma)$ induced by $(\widetilde{\mu_t}, t \geq 0)$. When $G = \mathbb{R}, \Gamma = \mathbb{Z}_+$ and f_t is the Gaussian heat kernel, then the trace formula coincides with the well–known Poisson summation formula. However we were unable to show this fact for the symmetric α–stable semigroups wherein $1 \leq \alpha < 2$. Indeed, a numerical example was presented for the Cauchy semigroup at the end of section 5 (the case $\alpha = 1$), which showed that the Poisson summation formula could not hold in this case. The purpose of this corrigendum is to explain why that calculation was incorrect, and to show that the Poisson summation formula does in fact holds in this case.

We first discuss some contextual background for the error that this note is intended to correct. Assume $f \in L^1(\mathbb{R}^d)$. The standard Fourier transform used by analysts is

$$\widehat{f}(y) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot y} f(x) dx,$$

for $y \in \mathbb{R}^d$. However probabilists instinctively use

$$\widehat{f}^{(P)}(y) = \int_{\mathbb{R}^d} e^{i x \cdot y} f(x) dx,$$

as this is consistent with the notion of characteristic function Φ_X of a random variable X taking values in $\mathbb{R}^d$. Indeed if X has a density f, then

$$\Phi_X(y) := \mathbb{E}(e^{i y \cdot X}) = \widehat{f}^{(P)}(y).$$

Usually these different conventions do not cause any problems, but in the numerical example at the very end of section 5 in [1], I concluded that the Poisson summation formula failed for the Cauchy distribution (with $d = 1$) wherein[1] $f(x) = \frac{1}{\pi(1+x^2)}$ by carrying out numerical calculations. The problem is that I used $\widehat{f}^{(P)}(y) = e^{-|y|}$, when I should have used $\widehat{f}(y) = e^{-2\pi|y|}$.

[1] In relation to the discussion in the opening paragraph $f := f_1$

Let us first present the calculations using the correct Fourier transform. We have

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n) = 1 + \frac{2}{e^{2\pi}-1} = 1.003742,$$

and summing 10000 terms indicates that
$\sum_{n=1}^{\infty} 1/(1+n^2)$ is approximately 1.076574, and so $\sum_{n\in\mathbb{Z}} f(n)$ is approximately 1.003678. This constitutes convincing evidence that the Poisson summation formula does hold in this case.

In fact, we can go further and prove that it holds by using the fact (see [2], top of page 155) that the Poisson summation formula is valid when both f and $\widehat{f}$ are of moderate decrease.[2] We only need to show this for $\widehat{f}$ as it is immediate for f. Since $\lim_{|x|\to\infty}(1+|x|^2)e^{-2\pi|x|} = 0$, given any $\epsilon > 0$, there exists $R > 0$ so that if $|x| > R$, we have $(1+|x|^2)e^{-2\pi|x|} < \epsilon$. Then

$$e^{-2\pi|x|} \leq \frac{A}{1+|x|^2},$$

where $A := \max\left\{\epsilon, \sup_{|x|\leq R}(1+|x|^2)e^{-2\pi|x|}\right\}$. The same argument works for arbitrary $t > 0$, so the Poisson summation formula is valid in that case, and takes the form:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi t|n|} = 1 + \frac{2}{e^{2\pi t}-1}.$$

This result seems to be well–known in the harmonic analysis community. It is Problem 19(a) on p.166 in [2].

We may now conclude that the Poisson summation formula for symmetric α–stable processes on $\mathbb{R}$ holds for $\alpha = 1$ and $\alpha = 2$. The problem remains open if $1 < \alpha < 2$. The same holds for rotationally invariant α–stable processes on $\mathbb{R}^d$, using similar arguments to the above for $\alpha = 1$.

References

[1] D.Applebaum, Probabilistic trace and Poisson summation formulae on locally compact Abelian groups, *Forum Math.*, to appear (2017) (DOI:10.1515/forum-2016-0067)

[2] E.M.Stein, R.Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press (2003)

[2] Here $\alpha = 1$ in relation to the definition (5.2) given in [1].