SECOND QUANTISATION FOR SKEW CONVOLUTION PRODUCTS OF INFINITELY DIVISIBLE MEASURES

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Abstract. Suppose $\lambda_1$ and $\lambda_2$ are infinitely divisible Radon measures on real Banach spaces $E_1$ and $E_2$, respectively and let $T : E_1 \to E_2$ be a Borel measurable mapping so that $T(\lambda_1) \ast \rho = \lambda_2$ for some Radon probability measure $\rho$ on $E_2$. Extending previous results for the Gaussian and the Poissonian case, we study the problem of representing the ‘transition operator’ $P_T : L^p(E_2, \lambda_2) \to L^p(E_1, \lambda_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y) d\rho(y)$$

as the second quantisation of a contraction operator acting between suitably chosen ‘reproducing kernel Hilbert spaces’ associated with $\lambda_1$ and $\lambda_2$.

1. Introduction

Let $E_i$ ($i = 1, 2$) be real Banach spaces equipped with Radon probability measures $\lambda_1$ and $\lambda_2$, respectively. A Borel measurable mapping $T : E_1 \to E_2$ is called a skew map for the pair $(\lambda_1, \lambda_2)$ if there exists a Radon probability measure $\rho$ on $E_2$ so that $\lambda_2$ is the convolution of $\rho$ with the image of $\lambda_1$ under the action of $T$:

$$T(\lambda_1) \ast \rho = \lambda_2.$$

In this case for each $1 \leq p < \infty$ we obtain a linear contraction $P_T : L^p(E_2, \lambda_2) \to L^p(E_1, \lambda_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y) d\rho(y).$$

Such constructions arise naturally in the study of Mehler semigroups, linear stochastic partial differential equations driven by additive Lévy noise, and operator self-decomposable measures (see, e.g., [4, 6, 7, 9, 10, 17]). In this context, the problem of “second quantisation” is to find a functorial manner of expressing $P_T$ in terms of $T$. The reason for this name is that the first work on this subject [5], within the context of Gaussian measures, exploited constructions that were similar to those that are encountered in the construction of the free quantum field from one-particle space (see e.g. [14]) wherein the nth chaos spanned by multiple Wiener-Itô integrals corresponds to the n-particle space within the Fock space decomposition.

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In our previous paper [3] we implemented this programme and constructed $P_T$ as the second quantisation of $T$ in the two cases where the measure $\lambda_i$ are Gaussian (generalising [5] and [13]), and are infinitely divisible measures of pure jump type (generalising [15]). In this article, we complete the programme by dealing with the case where the $\lambda_i$ are general infinitely divisible measures. Recall that a Radon probability measure $\lambda$ on $E$ is said to be infinitely divisible for each integer $n \geq 1$ there exists a Radon probability measure $\lambda_1/n$ whose $n$-fold convolution product equals $\lambda$:

$$\lambda_1/n \ast \cdots \ast \lambda_1/n = \lambda.$$ 

These measures $\lambda_1/n$ are unique.

It is well-known that an infinitely divisible Radon probability measure $\lambda$ on $E$ admits a unique representation as the convolution

$$\lambda = \delta_\xi \ast \gamma \ast \tilde{e}_s(\nu),$$

where $\delta_\xi$ is the Dirac measure concentrated at the point $\xi \in E$, $\gamma$ is a centred Gaussian Radon measure on $E$, and $\tilde{e}_s(\nu)$ is the generalised exponential of a Radon Lévy measure $\nu$ on $E$ as in [8, Theorem 3.4.20].

It is useful to rewrite (1.1) from the point of view of random variables, rather than measures. By [8, Theorem 2.39] there exists a semigroup of Radon probability measures $(\lambda_t)_{t \geq 0}$ such that $\lambda = \lambda_1$. By the celebrated Kolmogorov construction (see, e.g., [1, pp. 64–5]) we may construct an $E$-valued process $(X_t)_{t \geq 0}$ such that the law of $X_t$ is $\lambda_t$ for each $t \geq 0$. Using the Lévy-Itô decomposition of Riedle and van Gaans [16], for $t = 1$ we may then write

$$X_1 = \xi + Q + \int_E x \, d\Pi(x),$$

where $\xi \in E$ is as in (1.1), $Q$ is the covariance of $\gamma$, and $\Pi$ is a Poisson random measure whose intensity measure $\nu$ is a Lévy measure on $E$ and

$$\Pi(dx) := 1_{\{0 < \|x\| \leq 1\}} \hat{\Pi}(dx) + 1_{\{|\|x\| > 1\}} \Pi(dx),$$

with $\hat{\Pi}$ the compensated Poisson random measure,

$$\hat{\Pi}(B) := \Pi(B) - \nu(B).$$

In this description, the measure $\tilde{e}_s(\nu)$ is the law of $\int_E x \, d\Pi(x)$.

The data $\xi$, $\gamma$, $\nu$ are uniquely determined by $\lambda$. For more details we refer to [3, 12, 15, 16]. In what follows we shall write

$$\pi := \delta_\xi \ast \tilde{e}_s(\nu)$$

for brevity.

From [3], we know that we can effectively realise the second quantisation of skew maps of $\gamma$ in the symmetric Fock space $\Gamma(H_\gamma)$ of the reproducing kernel Hilbert space $H_\gamma$ of $\gamma$; by the Wiener-Itô chaos decomposition this space is isomorphic to $L^2(E, \gamma)$. To second quantise skew maps of $\pi$, we use the fact that a similar result holds if instead of the symmetric Fock space over $H_\gamma$, we consider the symmetric Fock space over $L^2(E, \nu)$; this is precisely the approach adopted by Peszat in [15].

The independence of $X_\gamma$ and $X_\pi$ then suggests that in order to unify these two approaches one should use the symmetric Fock space over $H_\gamma \oplus L^2(E, \nu)$. As we shall demonstrate in this paper, this intuition is correct.
We finish this introduction by fixing some notation. All vector spaces are real. Unless otherwise stated, Banach spaces are denoted by $E, F, \ldots$, and Hilbert spaces by $H$. The dual of a Banach space $E$ is denoted by $E^*$; the duality pairing between vectors $x \in E$ and $x^* \in E^*$ is written as $(x, x^*)$. Using the Riesz representation theorem, the dual of a Hilbert space $H$ will always be identified with $H$ itself. The Fourier transform of a Radon probability measure $\mu$ defined on $E$ is the mapping $\hat{\mu} : E^* \rightarrow \mathbb{C}$ for which
\[
\hat{\mu}(x^*) = \int_E e^{i \langle x, x^* \rangle} d\mu(x).
\]

2. Skew Convolution Products of Infinitely Divisible Measures

We fix two infinitely divisible Radon probability measures $\lambda_1$ and $\lambda_2$, on the Banach spaces $E_1$ and $E_2$ respectively. We furthermore assume that a Borel linear mapping $T : E_1 \rightarrow E_2$ is given. The main result of this section gives a necessary and sufficient condition in order that $T$ be skew with respect to the pair $(\lambda_1, \lambda_2)$.

We recall the Lévy-Khintchine decompositions $\lambda_i = \gamma_i \ast \pi_i$ of (1.1) and (1.2) (for $i = 1, 2$)

**Proposition 2.1.** Under these assumptions the following assertions are equivalent:

1. $T$ is skew with respect to $(\lambda_1, \lambda_2)$ with an infinitely divisible skew factor;
2. $T$ is skew with respect to both $(\gamma_1, \gamma_2)$ and $(\pi_1, \pi_2)$ with infinitely divisible skew factors.

If these equivalent conditions are satisfied, the skew factor $\rho$ in (1) and the skew factors $\rho_\gamma$ and $\rho_\pi$ in (2) are related by $\rho = \rho_\gamma \ast \rho_\pi$.

**Proof.** We begin by making the preliminary observation that if $\alpha$ and $\beta$ are measures on $E_1$, then their image measures under $T$ satisfy $T(\alpha \ast \beta) = (T\alpha) \ast (T\beta)$. We shall freely use the properties of infinitely divisible measures on Banach space as can be found in [8, 12].

$(2) \Rightarrow (1)$: From
\[
T\lambda_1 \ast (\rho_\gamma \ast \rho_\pi) = (T\gamma_1 \ast T\pi_1) \ast (\rho_\gamma \ast \rho_\pi) = (T\gamma_1 \ast \rho_\gamma) \ast (T\pi_1 \ast \rho_\pi) = \gamma_2 \ast \pi_2 = \lambda_2
\]
we infer that $T$ is skew for $(\lambda_1, \lambda_2)$ with skew factor $\rho_\gamma \ast \rho_\pi$. This measure, being the convolution of two infinitely divisible measures, is infinitely divisible.

$(1) \Rightarrow (2)$: By the Lévy-Khintchine decomposition theorem we have $\lambda_i = \delta_{\xi_i} \ast \gamma_i \ast \hat{e}_s(\nu_i)$ $(i = 1, 2)$ using the notation introduced before we have
\[
\lambda_1 \ast \lambda_2 = (\delta_{\xi_1} \ast \gamma_1 \ast \hat{e}_s(\nu_1)) \ast (\delta_{\xi_2} \ast \gamma_2 \ast \hat{e}_s(\nu_2)) = \delta_{\xi_1 + \xi_2} \ast (\gamma_1 \ast \gamma_2) \ast \hat{e}_s(\nu_1 + \nu_2).
\]

By the uniqueness part of [8, Theorem 3.4.20], this shows that the Gaussian factor of $\lambda_1 \ast \lambda_2$ equals $\gamma_1 \ast \gamma_2$.

Now suppose that $T\lambda_1 \ast \rho = \lambda_2$ with each of the measures $\lambda_1, \lambda_2$, and $\rho$ infinitely divisible. Then $T\lambda_1$ is infinitely divisible with $T\lambda_1 = \delta_{T\xi_1} \ast T\gamma_1 \ast T\hat{e}_s(\nu_1)$, and applying the remark of the previous paragraph to $T\lambda_1$ and $\rho$ we find that the Gaussian factor of $T\lambda_1 \ast \rho$ equals $T\gamma_1 \ast \eta$, where $\eta$ is the Gaussian factor of $\rho$. It follows that
\[
T\gamma_1 \ast \eta = \gamma_2,
\]
that is, $T$ is skew with respect to $(\gamma_1, \gamma_2)$ with Gaussian factor $\eta$. Taking Fourier transforms, this means that
\[
\hat{T}\gamma_1 \hat{\eta} = \hat{\gamma}_2.
\]
Finally, taking Fourier transforms in the original identity $T\lambda_1 \ast \rho = \lambda_2$ we obtain $\hat{T}_{\gamma_1} \hat{T}_{\pi_1} \hat{\rho} = \hat{\gamma}_2 \hat{\pi}_2$ or equivalently, utilising (2.1)

$$\hat{T}_{\pi_1} \left( \frac{\hat{T}_{\gamma_1} \hat{\rho}}{\hat{\gamma}_2} \right) = \hat{T}_{\pi_1} \hat{\eta} \hat{\rho} = \hat{T}_{\pi_1} \hat{\eta} \ast \rho = \hat{\pi}_2.$$ 

From this we see that $T$ is skew with respect to $(\pi_1, \pi_2)$, with skew factor $\eta \ast \rho$. □

It is not true in general that $\mu_1 \ast \mu_2 = \mu_3$ with $\mu_1$ and $\mu_3$ infinite divisible implies the infinite divisibility of $\mu_2$. The following counterexample (in the case $E = \mathbb{R}$) is due to Jan Rosiński who kindly permitted its inclusion here.

**Example 2.2 (Rosiński).** Consider the signed measure $\nu := 2\delta_1 + 2\delta_2 - \delta_3 + 2\delta_4 + 2\delta_5$, where $\delta_x$ is the usual Dirac mass at $x \in \mathbb{R}$. We claim that $\phi(t) := \exp \left( \int_0^\infty (e^{itx} - 1) d\nu(x) \right)$ is the characteristic function of some non-negative random variable $Z$. This random variable cannot be infinitely divisible. Indeed, if it were, $\nu$ would be its Lévy measure, which is impossible because a Lévy measure is non-negative and unique. Therefore, to complete a counterexample we need to show that $\phi$ is a characteristic function. Consider

$$e(\nu) := \sum_{n=0}^{\infty} \frac{\nu^* n^n}{n!}.$$ 

First we compute

$$\nu^*2 = 4\delta_2 + 8\delta_3 + 4\delta_5 + 17\delta_6 + 4\delta_7 + 8\delta_9 + 4\delta_{10}$$

and

$$\nu^*3 = 8\delta_3 + 24\delta_4 + 12\delta_5 + 8\delta_6 + 66\delta_7 + 54\delta_8 - \delta_9 + 54\delta_{10} + 66\delta_{11} + 8\delta_{12} + 12\delta_{13} + 24\delta_{14} + 8\delta_{15}.$$ 

We have

$$\nu^*2 \geq 0, \quad \nu + \frac{1}{8} \nu^*2 \geq 0, \quad \nu^*2 + c \nu^*3 \geq 0 \quad (0 \leq c \leq 1).$$

Hence

$$e(\nu) = \delta_0 + (\nu + \frac{1}{3} \nu^*2) + \frac{1}{6} (\nu^*2 + \nu^*3) + \sum_{n=2}^{\infty} \frac{\nu^{*2(n-1)}}{(2n)!} \ast \left( \nu^*2 + \frac{\nu^*3}{2n+1} \right).$$

Consequently, $e(\nu)$ is a finite non-negative measure on $\mathbb{Z}_+$ with $(e(\nu))(\mathbb{Z}_+) = e^{-7} e(\nu)$. Take $Z$ to be a random variable with distribution $e^{-7} e(\nu)$. Then the characteristic function of $Z$ equals $\phi$. Now let $X$ be a compound Poisson random variable, independent of $Z$, and with Lévy measure $\delta_3$. Then $X + Z$ is compound Poisson with Lévy measure $2\delta_1 + 2\delta_2 + 2\delta_4 + 2\delta_5$.

An interesting case where infinite divisibility of the skew factors is automatic occurs in the context of Mehler semigroups; we refer to [17] for the details.
3. Second Quantisation

Suppose $\lambda$ is an infinitely divisible Radon measure on a real Banach space $E$. Then we may write

$$\lambda = \gamma \ast \pi$$

with $\gamma$ a centred Gaussian Radon measure on $E$ and $\pi$ the distribution of a random variable of the form $\xi + \int_E x \, d\Pi(x)$ as explained in the introduction.

For functions $f \in L^2(\lambda)$ put

$$F_f(x, y) := f(x + y), \quad x, y \in E.$$  

Using the fact that $L^2(\gamma) \otimes L^2(\pi) = L^2(\gamma \times \pi)$ isometrically (with $\otimes$ indicating the Hilbert space tensor product) it is immediate to verify that

$$\|f\|_{L^2(\lambda)}^2 = \int_E \int_E |f(x + y)|^2 \, d\gamma(x) \, d\pi(y) = \|F_f\|_{L^2(\gamma) \otimes L^2(\pi)}^2.$$

As a result the mapping $f \mapsto F_f$ is a linear isometry from $L^2(\lambda)$ into $L^2(\gamma) \otimes L^2(\pi)$. This brings us to the setting with independence structure as discussed in [2]. Following that reference, formally we define a derivative operator acting with dense domain in $L^2(\gamma) \otimes L^2(\pi)$ by the formula

$$D := D_\gamma \otimes I + I \otimes D_\pi,$$

where we denote the ‘Gaussian’ and the ‘Poissonian’ derivatives with subscripts $\gamma$ and $\pi$, respectively. Recall from [3] that these are defined as follows. The Gaussian derivative is defined by

$$D_\gamma f(x) := \sum_{n=1}^N \partial_n g(\phi_{h_1}(x), \ldots, \phi_{h_N}(x)) \otimes h_n$$

for cylindrical functions $f = g(\phi_{h_1}, \ldots, \phi_{h_N})$, with $g \in C^1_b(\mathbb{R}^N)$ and $\phi : h \mapsto \phi_h$ being the isometry which embeds the reproducing kernel Hilbert space $H_\gamma$ of $\gamma$ onto the first Wiener-Itô chaos of $L^2(\gamma)$. The space of all such functions $f$ is dense in $L^2(\gamma)$ and $D_\gamma$ is closable as an operator from this initial domain into $L^2(\gamma; H_\gamma)$. The Poissonian derivative is defined by

$$D_\pi f(x) = f(x + \cdot) - f(x).$$

In order to prove that $D_\pi$ is densely defined as an operator from $L^2(\pi)$ into $L^2(\pi \times \nu)$ we need to find a dense set of functions $f$ in $L^2(\pi)$ such that $D_\pi f$ belongs to $L^2(\pi \times \nu)$. For this, we consider cylindrical functions $f$ of the form

$$f(x) = g(\langle x, x_1^* \rangle, \ldots, \langle x, x_N^* \rangle)$$

with $g \in C^1_b(\mathbb{R}^N)$ and $x_1^*, \ldots, x_N^* \in E^*$. For such $f$ we have, where $0 < \theta_n(\cdot) < 1$ for each $n \in \mathbb{N}$,

$$\|D_\pi f\|^2 = \int_{E \times E} \left| g(\langle x + y, x_1^* \rangle, \ldots, \langle x + y, x_N^* \rangle) - \sum_{n=1}^N \theta_n(\langle x, x_1^* \rangle, \ldots, \langle x, x_N^* \rangle) \right|^2 \, d\pi(x) \, d\nu(y)$$

$$= \int_{\{\|y\| > 1\} \times E} \left| g(\langle x + y, x_1^* \rangle, \ldots, \langle x + y, x_N^* \rangle) - \sum_{n=1}^N \theta_n(\langle x, x_1^* \rangle, \ldots, \langle x, x_N^* \rangle) \right|^2 \, d\pi(x) \, d\nu(y)$$
\[ + \int \sum_{n=1}^{N} (\partial_n g(\langle x + \theta_1(x)y, x_1^* \rangle, \ldots, \langle x + \theta_N(x)y, x_N^* \rangle)) \langle y, x_n^* \rangle^2 d\pi(x) \, d\nu(y) \]
\[
\leq 4 \|g\|_2^2 \nu(\|y\| > 1) + \sum_{n=1}^{N} \|\partial_n g\|^2_\infty \int_{\|y\| \leq 1} |\langle y, x_n^* \rangle|^2 \, d\nu(y) < \infty, \]
the finiteness in the last step being a consequence of the general properties of Lévy measures on Banach spaces (see \cite[pp. 95–120]{8} or \cite[pp. 69–75]{12}).

**Lemma 3.1.** \(D_\pi\) is closable as a densely defined linear operator from \(L^2(\pi)\) to \(L^2(\pi \times \nu)\).

**Proof.** Suppose \(f_n \to 0\) in \(L^2(\pi)\) and \(D_\pi f_n \to F\) in \(L^2(\pi \times \nu)\). We must prove that \(F = 0\). Passing to a subsequence, we may assume that \(f_n(x) \to 0\) for \(\pi\)-almost all \(x \in E\) and \(D_\pi f_n(x, y) = f_n(x + y) - f_n(x) \to F(x, y)\) for \(\pi \times \nu\)-almost all \((x, y) \in E \times E\). Then, by Fubini’s theorem, for \(\nu\)-almost all \(y \in E\) we have \(f_n(x+y) \to F(x,y)\) for \(\pi\)-almost all \(x \in E\). Since for all \(y \in E\) we have \(f_n(x+y) \to 0\) for \(\pi\)-almost all \(x \in E\), it follows that for \(\nu\)-almost all \(y \in E\) we have \(F(x,y) = 0\) for \(\pi \times \nu\)-almost all \((x,y) \in E \times E\).

From now on, we use the notations \(D_n\) and \(D_\pi\) for the closures of the operators considered so far and denote by \(D(D_n)\) and \(D(D_\pi)\) their domains.

**Lemma 3.2.** Suppose \(T_1 : E_1 \to F_1\) and \(T_2 : E_2 \to F_2\) are densely defined closed linear operators, with domains \(D(T_1)\) and \(D(T_2)\) respectively. Let \(G\) be another Banach space and let \(X \hat{\otimes} Y\) denote the completion of \(X \otimes Y\) with respect to any norm which has the property that \(\|x \otimes y\| = \|x\|\|y\|\) for all \(x \in X\) and \(y \in Y\).

1. The operators \(T_1 \otimes I : E_1 \hat{\otimes} G \to F_1 \hat{\otimes} G\) and \(I \otimes T_2 : G \hat{\otimes} E_2 \to G \hat{\otimes} F_2\) with their natural domains \(D(T_1) \otimes G\) and \(G \otimes D(T_2)\) are closable;
2. The operator \(T_1 \otimes I + I \otimes T_2 : E_1 \hat{\otimes} E_2 \to F_1 \hat{\otimes} F_2\) with its natural domain \(D(T_1) \otimes D(T_2)\) is closable.

**Proof.** Part (1) is immediate from the fact that \(\|x \otimes y\| = \|x\|\|y\|\); part (2) follows from the fact that a densely defined linear operator is closable if and only if its domain is weak*-densely defined, along with the operator inclusion \(T_1^* \otimes I + I \otimes T_2^* \subseteq (T_1 \otimes I + I \otimes T_2)^*\). The details are left to the reader. \(\square\)

To proceed any further we define, for \(n = 0, 1, 2, \ldots\), the Hilbert spaces
\[
\mathcal{H}_n := \bigoplus_{j,k \geq 0 \atop j + k = n} H_\gamma^\otimes j \hat{\otimes} L^2(\nu)^\otimes k.
\]

We use the convention that \(G^\otimes 0 = \mathbb{R}\) for any Hilbert space \(G\) and recall that \(\hat{\otimes}\) refers to the Hilbertian completion of the algebraic tensor product. We set
\[
\mathcal{H} := \mathcal{H}_1 = (H_\gamma \hat{\otimes} \mathbb{R}) \oplus (\mathbb{R} \hat{\otimes} L^2(\nu)) = H_\gamma \oplus L^2(\nu).
\]

Having defined \(D_\gamma\) (respectively \(D_\pi\)) as closed densely defined operators from \(L^2(\gamma)\) into \(L^2(\gamma) \hat{\otimes} H_\gamma\) (respectively from \(L^2(\pi)\) into \(L^2(\pi \times \nu) = L^2(\pi) \hat{\otimes} L^2(\nu)\)), we now identify both \(L^2(\gamma) \hat{\otimes} H_\gamma\) and \(L^2(\pi) \hat{\otimes} L^2(\nu)\) canonically with closed subspaces of \((L^2(\gamma) \hat{\otimes} L^2(\pi)) \hat{\otimes} (H_\gamma \oplus L^2(\nu)) = L^2(\gamma \times \pi; \mathcal{H})\). We denote by \(D_\gamma \otimes I\) and \(I \otimes D_\pi\)
the resulting closed and densely defined operators from $L^2(\gamma) \hat{\otimes} L^2(\pi) = L^2(\gamma \times \pi)$ into $L^2(\gamma \times \pi; \mathcal{H})$, and define

$$D = D_\gamma \otimes I + I \otimes D_\pi.$$  

By part (1) of the previous lemma, after completing we can consider $D_\gamma \otimes I$ and $I \otimes D_\pi$ as closed and densely defined operators from $L^2(\gamma \times \pi; \mathcal{H}_n)$ into $L^2(\gamma \times \pi; \mathcal{H}_{n+1})$.

By combining the preceding two lemmas we obtain the following result.

**Proposition 3.3.** For all $n = 0, 1, 2, \ldots$, the operator $D = D_\gamma \otimes I + I \otimes D_\pi$ is closable as a densely defined operator from $L^2(\gamma \times \pi; \mathcal{H}_n)$ into $L^2(\gamma \times \pi; \mathcal{H}_{n+1})$.

We define the $n$-fold stochastic integral on $I_n : \mathcal{H}_n \to L^2(\Omega)$ by

$$I_n(f \otimes g) := I_{j,\gamma}f \otimes I_{k,\pi}g$$

for $f \in H^j$ and $g \in L^2(\nu)^k$ with $j + k = n$, where we denote the ‘Gaussian’ and the ‘Poissonian’ integrals with subscripts $\gamma$ and $\pi$, respectively.

In what follows, in order to tidy up the notation we will refrain from writing subscripts $\gamma$ and $\pi$: expectations taken in the the left and right sides of tensor products refer to $\gamma$ and $\pi$, respectively.

Let $\Pi$ be a Poisson random measure on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose intensity measure $\nu$ is a Lévy measure on $E$. Recall that the former means that $\Pi$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in the space $\mathbb{N}(E)$ of $\mathbb{N}$-valued measures on $E$ endowed with the $\sigma$-algebra generated by the Borel sets of $E$, that is, the smallest $\sigma$-algebra which renders the mappings $\xi \mapsto \xi(B)$ measurable for all $B \in \mathcal{B}(E)$. By $\mathbb{P}_\Pi$ we denote the image measure of $\mathbb{P}$ under $\Pi$.

Following Last and Penrose [11], for a measurable function $f : \mathbb{N}(Y) \to \mathbb{R}$ and $y \in Y$ we define the measurable function $\hat{D}_y f : \mathbb{N}(Y) \to \mathbb{R}$ by

$$\hat{D}_y f(\eta) := f(\eta + \delta_y) - f(\eta).$$

The function $\hat{D}^n_{y_1, \ldots, y_n} f : \mathbb{N}(Y) \to \mathbb{R}$ is defined recursively by

$$\hat{D}^n_{y_1, \ldots, y_n} f = \hat{D}_{y_n} \hat{D}^{n-1}_{y_1, \ldots, y_{n-1}} f,$$

for $y_1, \ldots, y_n \in Y$. This function is symmetric, i.e. it is invariant under any permutation of the variables.

Following [3], we define $j : L^2(E, \mu) \to L^2(\mathbb{P}_\Pi)$ by

$$j f(\eta) = f\left(\xi + \int_E x \eta(dx)\right), \quad \eta \in \mathbb{N}(E).$$

The rigorous interpretation of this identity is provided by noting that

$$\|j f\|^2_{L^2(\mathbb{P}_\Pi)} = E\left|f\left(\xi + \int_E x d\Pi(x)\right)\right|^2 = \|f\|^2_{L^2(E, \mu)},$$

which means that $j f(\eta)$ is well-defined for $\mathbb{P}_\Pi$-almost all $\eta$ and that $j$ establishes an isometry from $L^2(E, \mu)$ into $L^2(\mathbb{P}_\Pi)$. Note that

$$j f(\Pi) = f\left(\xi + \int_E x d\Pi(x)\right)$$

and

$$j \circ D = \hat{D} \circ j.$$
We now have the following extension to infinitely divisible measures of the corresponding results of Stroock [18] (for Gaussian measures) and Last and Penrose [11] (for Poisson random measures):

**Proposition 3.4.** For all \( f \in W^{\infty,2}(\gamma) \) and \( g \in L^2(\mathbb{P}_\Pi) \) we have

\[
f \otimes g(\Pi) = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E}(\tilde{D}^m f \otimes g(\Pi))).
\]

**Proof.** By Leibniz’s rule,

\[
\sum_{m=0}^{\infty} \frac{1}{m!} I_m \mathbb{E}_\lambda \tilde{D}^m f \otimes g(\Pi) = \sum_{m=0}^{\infty} \frac{1}{m!} I_m \mathbb{E}_\lambda \left( \sum_{\ell=0}^{m} \binom{m}{\ell} \tilde{D}^\ell f \otimes \tilde{D}^{m-\ell} g(\Pi) \right)
\]

\[
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_m \mathbb{E}_\lambda \left( \tilde{D}^\ell f \otimes \tilde{D}^{m-\ell} g(\Pi) \right)
\]

\[
= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_\ell(\mathbb{E}_\lambda \tilde{D}^\ell f) \otimes I_{m-\ell}(\mathbb{E}_\pi \tilde{D}^{m-\ell} g(\Pi))
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{j!} I_j(\mathbb{E}_\gamma \tilde{D}^j f) \otimes \sum_{k=0}^{\infty} \frac{1}{k!} I_k(\mathbb{E}_\pi \tilde{D}^k g(\Pi))
\]

\[
= f \otimes g(\Pi).
\]

using the Stroock and Last-Penrose type decompositions in the penultimate identity.

We now return to the setting considered in Section 2 and make the standing assumption that the equivalent conditions stated in Proposition 2.1 are satisfied. Thus we assume that \( \lambda_1 = \gamma_1 * \pi_1 \) on \( E_1 \), \( \lambda_2 = \gamma_2 * \pi_2 \) on \( E_2 \), and that \( T : E_1 \to E_2 \) is a Borel linear skew mapping with respect to the pair \( (\lambda_1, \lambda_2) \) with an infinite divisible skew factor. As is shown by Proposition 2.1, this implies that \( T \) is skew with respect to both pairs \( (\gamma_1, \gamma_2) \) and \( (\pi_1, \pi_2) \), that is, \( T\gamma_1 * \rho_\gamma = \gamma_2 \) and \( T\pi_1 * \rho_\pi = \pi_2 \).

It follows Proposition 2.1 that we may define \( P_T : L^2(E_2, \lambda_2) \to L^2(E_1, \lambda_1) \) by

\[
P_T f(x) := \int_{E_2} f(Tx + y) \, d\rho(y), \quad x \in E_1,
\]

where \( \rho := \rho_\gamma * \rho_\pi \) is the skew factor on \( E_2 \), i.e., \( T\lambda_1 * \rho = \lambda_2 \). Similarly we can define an operator \( P_T \otimes P_T : L^2(\gamma_2) \otimes L^2(\pi_2) \to L^2(\gamma_1) \otimes L^2(\pi_1) \) in the obvious way (with slight abuse of notation; we should really be writing \( P_{\gamma,T} \otimes P_{\pi,T} \)) and we then have:

**Lemma 3.5.** Under the above assumptions, \( F_{P_T} = (P_T \otimes P_T)F_f \).

**Proof.** For \( (\gamma \times \pi) \)-almost all \( x, y \in E_2 \) we have

\[
(P_T \otimes P_T)(\phi \otimes \psi)(x, y) = (P_{T\phi} \otimes P_{T\psi})(x, y)
\]

\[
= \int_{E_2} \phi(Tx + z) \, d\rho_\gamma(z) \int_{E_2} \psi(Ty + z) \, d\rho_\pi(z)
\]

\[
= \int_{E_2} \int_{E_2} (\phi \otimes \psi)(Tx + z_1, Ty + z_2) \, d\rho_\gamma(z_1) \, d\rho_\pi(z_2).
\]
Now suppose that $F = \lim_{n \to \infty} G_n$ in $L^2(\gamma \times \pi)$, where each $G_n$ belongs to the algebraic tensor product $L^2(\gamma) \otimes L^2(\pi)$. By the above identity and linearity it follows, after passing to a subsequence if necessary, that for $(\gamma \times \pi)$-almost all $x, y \in E_2$ we have

\[
(P_T \otimes P_T) F_{f}(x, y) = \lim_{n \to \infty} (P_T \otimes P_T) G_n(x, y)
= \lim_{n \to \infty} \int_{E_2} \int_{E_2} G_n(Tx + z_1, Ty + z_2) \, d\rho_\gamma(z_1) \, d\rho_\pi(z_2)
= \int_{E_2} \int_{E_2} F_f(Tx + z_1, Ty + z_2) \, d\rho_\gamma(z_1) \, d\rho_\pi(z_2)
= \int_{E_2} \int_{E_2} f(Tx + Ty + z_1 + z_2) \, d\rho_\gamma(z_1) \, d\rho_\pi(z_2)
= \int_{E_2} \int_{E_2} f(Tx + Ty + z) \, d(\rho_\gamma * \rho_\pi)(z)
= \int_{E_2} \int_{E_2} f(Tx + Ty + z) \, d\rho(z)
= P_T f(x + y)
= F_{P_T f}(x, y).
\]

□

For $h \in H_\gamma$ and $y_1, \ldots, y_n \in E$ and $h \in H_\gamma$ we define

\[
D_{h; y} := D_h \otimes I + I \otimes D_y,
\]

where

\[
D_h f(x) := (D_\gamma f(x), h), \quad D_y g(x) := (D_\pi g(x))(y).
\]

For the higher order derivatives we define inductively

\[
D^n_{h_1, \ldots, h_n; y_1, \ldots, y_n} := D_{h_n; y_n} D^{n-1}_{h_1, \ldots, h_{n-1}; y_1, \ldots, y_{n-1}}.
\]

Lemma 3.6. For all $f \in L^2(E_2, \lambda_2)$, $h \in H_\gamma$, and $y_1, \ldots, y_n \in E_1$,

\[
\mathbb{E}_{\gamma_1 \times \pi_1} D^n_{h_1, \ldots, h_n; y_1, \ldots, y_n} F_{P_T f} = \mathbb{E}_{\gamma_2 \times \pi_2} D^n_{T h_1, \ldots, T h_n; Ty_1, \ldots, Ty_n} F_f.
\]

Proof. We approximate $F_f$ by finite sums of elementary tensors as in the proof of the previous lemma. For such functions $G_n$ the identity follows from the results in [3] for the Gaussian and Poissonian case. Thanks to the closedness of the derivative operators, the identity passes over to the limit. □

For Hilbert spaces $H$ and $H$ we note that

\[
\Gamma(H \oplus H) = \bigoplus_{n=0}^\infty \left( \bigoplus_{j,k \geq 0} H^{\otimes j} \hat{\otimes} H^{\otimes k} \right).
\]

Putting everything together we obtain the following result which generalises the results of Theorems 3.5 and 4.4 of [3], where Gaussian and Poisson noises were treated separately.
Theorem 3.7. Under the standing assumption stated above, the following diagram commutes:

\[
\begin{array}{ccc}
L^2(E_2, \lambda_2) & \xrightarrow{P_t} & L^2(E_1, \lambda_1) \\
\xrightarrow{f \mapsto F_t} & & \xrightarrow{f \mapsto F_t} \\
L^2(E_2 \times E_2, \gamma_2 \times \pi_2) & \xrightarrow{P_t \otimes P_t} & L^2(E_1 \times E_1, \gamma_1 \times \pi_1) \\
\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_2 D^n & \xrightarrow{\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_1 D^n} & \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_1 D^n \\
\Gamma((H_{\gamma,2} \otimes L^2(E_2, \nu_2)) & \xrightarrow{\bigoplus_{n=0}^{\infty} (T^n)^{\otimes n}} & \Gamma(H_{\gamma,1} \otimes L^2(E_1, \nu_1))
\end{array}
\]

Moreover, for \( k = 1, 2 \) also the following diagram commutes in distribution if \( X_k \) is an \( E \)-valued random variable with distribution \( \lambda_k \):

\[
\begin{array}{ccc}
L^2(E_k \times E_k, \gamma_k \times \pi_k) & \xrightarrow{(f,g) \mapsto (f(X_{\gamma,k})f(X_{\pi,k}))} & L^2(\Omega \times \Omega) \\
\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_k D^n & \xrightarrow{\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_k D^n} & \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_k D^n \\
\Gamma(H \otimes L^2(E_k, \nu_k)) & \xrightarrow{\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n}} E_{n} \times \pi_k D^n} & \Gamma(H \otimes L^2(E_k, \nu_k))
\end{array}
\]

References

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