*Metric Spaces - Additional Examples*

Metric spaces turn up all over mathematics. Here's some interesting examples.

1. **Coding** A *codeword* is a list of 0s and 1 of the form $a_1a_2\ldots a_n$. The number $n$ is called the *length* of the codeword e.g. suppose that $x = 010001, y = 100111$. In these two cases $n = 5$. The *Hamming distance* between two codewords of the same length is the number of places where they differ. So for the codewords considered above $d(x, y) = 4$. If $a = 110011$ and $b = 100011$ then $d(a, b) = 1$. It can be shown that $d$ is a metric indeed (M1) and (M2) are easy. (M3) is challenging. The Hamming distance has a practical use in describing transmission errors when the codeword received is different to that sent out.

2. **Probability**

   Let $S = \{\omega_1, \omega_2 \ldots, \omega_N\}$ be a finite set. A *probability* on $S$ is set of numbers $\{p(\omega_1), p(\omega_2), \ldots, p(\omega_n)\}$ which have the property that $p(\omega_i) > 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} p(\omega_i) = 1$. We can think of the points in $S$ as being the possible outcomes of some random procedure and $p(\omega_i)$ is the probability that $\omega_i$ occurs. From now on I'll write $p_i$ instead of $p(\omega_i)$ for simplicity.

   A *random variable* $X$ on $S$ is a mapping from $S$ to $\mathbb{R}$ and the range of $X$ are the numbers $x_1 = X(\omega_1), x_2 = X(\omega_2), \ldots, x_n = X(\omega_n)$. The *expectation* of $X$ is

$$\mathbb{E}(X) = \sum_{i=1}^{n} x_i p_i.$$

   If $X$ and $Y$ are two random variables defined on $S$ then the distance between them is

$$\begin{aligned} d(X, Y) &= \mathbb{E}(|X - Y|) \\ &= \sum_{i=1}^{n} |x_i - y_i| p_i. \end{aligned}$$

   In this case it is fairly straightforward to prove that $d$ is a metric on the set $\mathcal{R}(S)$ of all random variables on $S$.

3. **Special Relativity**

Special relativity is a radical new way of thinking about space and time that was introduced by Albert Einstein in 1905. It is now an established part of modern physics. Einstein's teacher Hermann Minkowski gave a lecture in 1908 from which the following quite famous quote is taken:

*"The views of space and time that I wish to lay before you have sprung from the soil of experimental physics and therein lies their strength. They are radical. Henceforth space by itself and time by itself are doomed to fade away into mere shadows and only a kind of union of the two will preserve an independent reality."*

Space-time provides us with an example of a "distance function" that is NOT a metric. Space-time is the set of all points $(x_1, x_2, x_3, x_4)$ where $(x_1, x_2, x_3)$ are the usual co-ordinates in three-dimensional space and $x_4 = ict$. Here $i = \sqrt{-1}$, $c$ is the velocity of light (in a vacuum) and $t$ is a time co-ordinate.

From a physical point of view the "natural" distance between two points $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$ (where $x_4 = ict_1$ and $y_4 = ict_2$) is

$$\begin{aligned} & \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 + (y_4 - x_4)^2} \\ = & \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 + (ict_2 - ict_1)^2} \\ = & \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 - c^2(t_2 - t_1)^2}, \end{aligned}$$

and this will be an imaginary number whenever

$$(t_2 - t_1)^2 > \frac{1}{c^2}[(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2].$$