A Lévy-Cielsielski Expansion for Quantum Brownian Motion and the Construction of Quantum Brownian Bridges

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Dedicated to Robin Hudson on his 65th birthday.

Abstract

We introduce "probabilistic" and "stochastic Hilbertian structures". These seem to be a suitable context for developing a theory of "quantum Gaussian processes". The Schauder system is utilised to give a Lévy-Cielsielski representation of quantum Brownian motion as operators in Fock space over a space of square summable sequences. Quantum Brownian bridges are defined and a number of representations of these are given.

Keywords and phrases: daggered space, probabilistic Hilbertian structure, stochastic Hilbertian structure, Fock space, exponential vector, quantum Brownian motion, Haar system, Schauder system, Lévy-Cielsieski expansion, quantum Brownian bridge.

MSC 2000 81S25, 60H99, 42C40, 81S05

1 Introduction

This paper falls naturally into two linked but distinct parts.The first part addresses itself to to finding a mathematical framework for "quantum random variables" and "quantum stochastic processes". The usual theory of such processes was established in a well-known paper by Accardi, Frigerio and Lewis ([1]). At its heart is the notion of a quantum random variable as an algebra homomorphism from a "state space algebra" into a "probability space algebra." This generalises the fact that every classical random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a measurable space $(U, \mathcal{U})$ gives rise to a homomorphism j from the algebra $B_b(U)$ of bounded measurable functions on U into $L^\infty(\Omega, \mathcal{F}, P)$, given by $j(f) = f \circ X$, for each $f \in B_b(U)$.

On the other hand, earlier in 1977, Cockcroft and Hudson [8] defined quantum Brownian motion to be a certain family $(Q(t), P(t), t \geq 0)$ of pairs of non-commuting self-adjoint operators acting in a Hilbert space which is equipped with a distinguished unit vector to determine expectations. Each of the processes $(Q(t), t \geq 0)$ and $(P(t), t \geq 0)$ is unitarily equivalent to a classical Brownian motion, although the pair cannot be simultaneously diagonalised. Quantum Brownian motion is not a quantum stochastic process in the sense of [1]. For most workers in the field, this has never been a serious problem. The Fock space version of quantum Brownian motion is one of the basic noises for the highly successful theory of quantum stochastic calculus. For example, it is used to set up quantum stochastic differential equations whose solutions are quantum Markov processes in the sense of [1] (for textbook accounts see [17] and [21]; [3] and [14] are more recent surveys).

The last part of this paper introduces quantum Brownian bridges. Like quantum Brownian motion, they are not quantum stochastic processes in the sense of [1] - they appear as natural transformations of the quantum Brownian motion operators in direct analogy with the classical theory. This was the motivation to develop an abstract formalism in which both quantum Brownian motion and quantum Brownian bridges will fit naturally. This is carried out in the first part of this paper. The setting is spatial rather than algebraic (c.f. [9]). The two main concepts are called probabilistic Hilbertian systems (to describe "quantum random variables") and stochastic Hilbertian systems (for "quantum stochastic processes "). Classical Gaussian fields fit naturally into this framework. Indeed it may well be that our formalism is the appropriate one for the development of a general theory of "quantum Gaussian processes" while the Accardi-Lewis-Frigerio one is more apposite for "quantum Markov processes."

In the second part of this paper we turn our attention to quantum Brownian motion. Classical Brownian motion has a delightful wavelet expansion obtained by combining the Schauder system with a sequence of i.i.d. standard normals. A beautiful recent textbook account is given in [23] (see also [15] for an older version before the language of wavelets was in vogue). The idea of constructing Brownian motion in this way was originally due to Paul Lévy and later, Z.Cielsielski (see also the comments on pp.18-19 of [18]). The main technical result of this paper is to obtain a quantum version of this expansion and so construct quantum Brownian motion in Fock space over $l^2(\mathbb{Z}_+)$. Consequently, only the discrete skeleton provided by a "quantum random walk" is required to generate the continuous time process. Our result seems easier to establish than the classical one of Lévy-Cielsielski as we don't require logarithmic growth estimates on the squares of i.i.d. Gaussians, thanks to the nice action of annihilation operators on exponential vectors. We do however require a more elaborate technology.

In the last part of this paper, we define and construct quantum Brownian bridges as discussed above. It is perhaps rather surprising that this hasn't been done before, given that it is such a natural step from quantum Brownian motion. We emphasise again that within our formalism of stochastic Hilbertian systems, it is easy to see that these are natural quantum Gaussian processes. It is not clear how this would be done within the standard theory of [1].

Acknowledgements It is a pleasure to thank Martin Lindsay for inviting me to give a talk at a conference at Lancaster University in October 2005 to celebrate Robin Hudson's 65th birthday. This proved to be the spur to develop the ideas on which this paper is based. I would also like to thank Nick Bingham for some helpful observations.

2 Probabilistic and Stochastic Hilbertian Structures

Let H be a complex separable Hilbert space and $\mathcal{D}$ be a dense linear subspace in H. We will be interested in linear operators T defined on H which have the following properties.

(i) $\mathcal{D} \subseteq \mathrm{Dom}(T)$ and the restriction of T to $\mathcal{D}$ is closable.

(ii) $\mathcal{D} \subseteq \mathrm{Dom}(T^*)$.

(iii) $T\mathcal{D} \subseteq \mathcal{D}$.

(iv) $\mathrm{Ran}(T^\dagger) \subseteq \mathrm{Dom}(T)$, where $T^\dagger$ denotes the restriction of T^* to $\mathcal{D}$.

In the sequel, we will often employ the notation $T^{\#}$ to mean T or $T^\dagger$.

Let $T_1, \cdots, T_n$ be linear operators satisfying (i) to (iv) above. We denote by $\mathcal{L}_n(\mathcal{D})$ the complex linear space generated by $\{T_1, \cdots, T_n, T_1^\dagger, \ldots, T_n^\dagger\}$. As we make no linear independence assumption, we have $\dim(\mathcal{L}_n(\mathcal{D})) \leq 2n$. We call $\mathcal{L}_n(\mathcal{D})$ a *daggered space* of order n. Slightly abusing terminology, $\{T_1, \ldots, T_n\}$ are called the *generators* of this space. Such a space is said to be *symmetric* if $\sum_{j=1}^{n}(\alpha_j T_j^\dagger - \overline{\alpha_j} T_j)$ is essentially skew-adjoint on $\mathcal{D}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. We define

$$U(\alpha) := \exp\left\{\left(\sum_{j=1}^{n}(\alpha_j T_j^\dagger - \overline{\alpha_j} T_j)\right)^c\right\}$$

to be the associated unitary operator in H (where ^{c} denotes closure).

Suppose that we are given two daggered spaces with disjoint generating sets. Their *sum* is the daggered space obtained by taking the complex linear space generated by the union of the two generating sets. If both spaces are symmetric, they are said to be *symmetrically summable* if their sum is also symmetric. A *probabilistic Hilbertian structure of order n* or *PHS(n)* is a pair $(\mathcal{L}_n(\mathcal{D}), \rho)$ where $\mathcal{L}_n(\mathcal{D})$ is a symmetric daggered space and ρ is a density operator in H, i.e a positive self-adjoint trace class operator with $\mathrm{tr}(\rho) = 1$. In the case where ρ is a pure state, i.e. the projection onto the ray generated by a unit vector ψ, we will abuse notation to the effect of denoting ρ by ψ. The *characteristic element* of a PHS(n) is the mapping $\phi : \mathbb{C}^n \to \mathbb{C}$ given by

$$\phi(\alpha) = \mathrm{tr}(\rho U(\alpha)).$$

Two PHS(n)s are said to be *identically distributed* if they have the same characteristic element. A PHS(n) is said to be of *classical type* if all its generators are essentially self-adjoint on $\mathcal{D}$. Two PHS(n)s, $(\mathcal{L}_m^{(i)}(\mathcal{D}_i), \rho_i)$ in $H_i (i = 1, 2)$, are said to be *equivalent* if there is a unitary isomorphism U from H_1 to H_2 for which $U\rho_1 U^{-1} = \rho_2$ and U intertwines the respective generators. They are *strongly equivalent* if they are equivalent and we also have $U\mathcal{D}_1 = \mathcal{D}_2$.

A *subsystem* of a PHS(n) $(\mathcal{L}_n(\mathcal{D}), \rho)$ is a PHS(m) $(\mathcal{L}_m(\mathcal{D}), \rho)$ with $m \leq n$ such that $\mathcal{L}_m(\mathcal{D}) \subseteq \mathcal{L}_n(\mathcal{D})$.

Let $(\mathcal{L}_m^{(1)}(\mathcal{D}), \rho)$ be a PHS(m) with characteristic element $\phi_m^{(1)}$ and $(\mathcal{L}_n^{(2)}(\mathcal{D}), \rho)$ be a PHS(n) with characteristic element $\phi_n^{(2)}$. Suppose that the associated daggered spaces are symmetrically summable. Then we obtain a new PHS($m + n$) which we denote as $(\mathcal{L}_{m+n}^{(1)+(2)}(\mathcal{D}), \rho)$, with characteristic element $\phi_{m+n}^{(1)+(2)}$. We call this the *sum* of $(\mathcal{L}_m^{(1)}(\mathcal{D}), \rho)$ and $(\mathcal{L}_n^{(2)}(\mathcal{D}), \rho)$. The two summands are said to be *independent* if

$$\phi_{m+n}^{(1)+(2)}(\alpha) = \phi_m^{(1)}(\pi(\alpha))\phi_n^{(2)}((I - \pi)\alpha),$$

for all $\alpha \in \mathbb{C}^{m+n}$, where π is the orthogonal projection from $\mathbb{C}^{m+n}$ to $\mathbb{C}^m$ which leaves the first m components of every vector invariant and maps the remaining n to zero.

A PHS(n) is said to be *Gaussian* with if there exists $m \in \mathbb{C}^n$ and an $n \times n$ positive definite symmetric matrix C such that

$$\phi(\alpha) = \exp\left\{-\frac{1}{2}(\overline{\alpha - m})^T C(\alpha - m)\right\}.$$

If C is a multiple of the identity, we say that the PHS is *i.i.d. Gaussian.* The probabilistic motivation for this is that in this case ϕ is the product of n copies of the characteristic element of a fixed Gaussian PHS(1).

Example 1 (Gaussian Vectors)

Let $X = (X_1, \ldots, X_n)$ be a multivariate Gaussian random vector with mean $m \in \mathbb{R}^n$ and covariance matrix A. Take H to be $L^2(\mathbb{R}^n, \mu_X; \mathbb{C})$, where μ_X is the law of X. For each $n \in \mathbb{N}$, let h_n be the Hermite polynomial of degree n and for $1 \leq i \leq n$ define $h_n^i \in H$ by $h_n^i = h_n \circ \pi^i$ where $\pi_i : \mathbb{R}^n \to \mathbb{R}$ picks out the ith component of each vector in $\mathbb{R}^n$. We take $\mathcal{D}$ to be the linear span of all finite products of distinct h_n^is and each T_i to be $\frac{1}{2}M_{X_i}$, where $(M_{X_i}f)(x) = \pi^i(x)f(x)$, for each $x \in \mathbb{R}^n$. Let $\psi_0 = 1$, then $(\{T_1, \ldots, T_n\}, 1)$ is a Gaussian PHS(n) with each

$$\phi(\alpha) = \exp\left\{-\frac{1}{2}(y - m)^T A(y - m)\right\},$$

where $y = (y_1, \ldots, y_n)$ with each $y_i = \Im(\alpha_i)$. In this case $m \in \mathbb{R}^n$ and identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the real vector space isomorphism which maps each $z \in C^n$ to $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, where each $z_j = x_j + iy_j (1 \leq j \leq n)$, we have $C = \begin{pmatrix} 0 & 0 \\ 0 & -A \end{pmatrix}$.

Example 2 (Quantum Harmonic Oscillators)

Here we take $H = \Gamma(\mathbb{C}^n)$ to be boson Fock space over $\mathbb{C}^n, \mathcal{D}$ to be the linear span of the finite particle vectors and a_i and $a_i^\dagger$ to be the annihilation and creation operators associated to the natural basis vector $e_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0)$. We consider the thermal state at inverse temperature $\beta > 0$ given by $\rho_\beta = \frac{1}{Z(\beta)} e^{-\beta H}$, where $H = \sum_{i=1}^n \omega_i a_i^\dagger a_i$ is the harmonic oscillator Hamiltonian with frequencies $\omega_i \geq 0$ $(1 \leq i \leq n)$ and

$$Z(\beta) = \mathrm{tr}(e^{-\beta H}) = \prod_{i=1}^n \frac{e^{-\frac{1}{2}\beta\omega_i}}{1 - e^{-\beta\omega_i}}.$$

It follows from [19], Chapter 12, section 12 (see also [9]) that $(\{a_1, \ldots, a_n\}, \psi_0)$ is a Gaussian PHS(n), with $m = 0$ and $C_{ij} = \coth\left(\frac{\omega_i}{2\beta}\right)\delta_{ij}, 1 \leq i, j \leq n$

Let $\mathcal{I}$ be an index set and $\{X(t), t \in \mathcal{I}\}$ be a family of linear operators in H. We call the pair $(\{X(t), t \in \mathcal{I}\}, \rho)$ a *stochastic Hilbertian structure* or *SHS* if for each $n \in \mathbb{N}$ and for each $t_1, \ldots, t_n \in \mathcal{I}$, $\{X(t_1), \ldots, X(t_n)\}$ generate a symmetric daggered space with respect to $\mathcal{D}$ of order n. This latter space is denoted $\mathcal{L}_{t_1,\ldots,t_n}(\mathcal{D})$. The collection of all $(\mathcal{L}_{t_1,\ldots,t_n}(\mathcal{D}), \rho)$s are called the *finite-dimensional distributions* of the SHS, by analogy with classical probability theory. Notions of equivalence and subsystem for SHSs are direct analogues of the definitions for PHS(n)s.

A SHS is said to be *Gaussian* if all of its finite-dimensional distributions are Gaussian.

Example 3 (Gaussian Probability Spaces [13, 16])

Let $(\Omega, \mathcal{F}, P; \mathcal{H})$ be a Gaussian probability space, so $\mathcal{H}$ is a real separable Hilbert space and there exists an isometric embedding Y of $\mathcal{H}$ into $L^2(\Omega, \mathcal{F}, P)$ such that for each $f \in \mathcal{H}$, $Y(f)$ is Gaussian with

$$\mathbb{E}(Y(f)) = 0, \quad \mathbb{E}(Y(f)Y(g)) = \langle f, g\rangle_{\mathcal{H}}.$$

We assume that this space is irreducible in the sense that $\mathcal{F} = \sigma\{\mathcal{G} \cup \mathcal{N}\}$, where $\mathcal{N}$ is the class of all P-null sets in Ω and $\mathcal{G} = \sigma\{Y(f), f \in \mathcal{H}\}$. Let $(e_n, n \in \mathbb{N})$ be an orthonormal basis in $\mathcal{H}$, and define $h_n^m = h_n \circ Y(e_m)$ for each $m \in \mathbb{N}$. The linear span of finite products of distinct h_n^ms are dense in $L^2(\Omega, \mathcal{F}, P)$. We take $\mathcal{D}$ to be the complexification of this space and H to be the complexification of $L^2(\Omega, \mathcal{F}, P)$. If we define $T_f = \frac{1}{2}M_f$, for each $f \in \mathcal{H}$, then $(\{Y(f), f \in \mathcal{H}\}, 1)$ is a Gaussian SHS, by similar arguments to those given in Example 1.

Note that classical and abstract Wiener spaces and also white noise spaces all fit into this context (see [13], pp.60-61).

Example 4 (Quantum Wiener Integrals)

Let $\Gamma(H)$ denote boson Fock space over H, $\mathcal{D}$ be the linear span of all the finite particle vectors in $\Gamma(H)$ and ψ_0 be the vacuum vector in $\Gamma(H)$. For each $f \in H, a(f)$ is the annihilation operator corresponding to f, $a^\dagger(f)$ is the creation operator and $W(f)$ is the Weyl operator (see e.g. [17], [21]) for full accounts of relevant "Focklore").

Take $H = L^2(\mathbb{R}^+)$ and fix $f \in H$.

Define $X(t) := a(f1_{[0,t)})$ for each $t \geq 0$. We note that, using the language of quantum stochastic integrals (see e.g. [12], [17], [21]):

$$X(t) = \int_0^t \overline{f(s)} dA(s), \quad X(t)^\dagger = \int_0^t f(s) dA^\dagger(s). \tag{2.1}$$

To see that $(X(t), t \in \mathbb{R}), \psi_0)$ is a Gaussian *SHS*, let $t_1, \dots t_n \in \mathbb{R}^+$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$; then

$$\begin{aligned}
\phi(\alpha) &= \left\langle \psi_0, \exp\left\{ \left(\sum_{j=1}^n [\alpha_j a^\dagger(f1_{[0,t_j)}) - \overline{\alpha_j} a(f1_{[0,t_j)})] \right)^c \right\} \psi_0 \right\rangle \\
&= \left\langle \psi_0, W\left(\sum_{j=1}^n \alpha_j f1_{[0,t_j)} \right) \psi_0 \right\rangle \\
&= \exp\left\{ -\frac{1}{2} \left|\left| \sum_{j=1}^n \alpha_j f1_{[0,t_j)} \right|\right|^2 \right\} \\
&= \exp\left\{ -\frac{1}{2} \bar{\alpha}^T C \alpha \right\},
\end{aligned}$$

where $C_{ij} = \displaystyle\int_0^{t_i \wedge t_j} |f(s)|^2 ds$ for each $1 \leq i, j \leq n$.

The above construction generalises in a natural way to the effect that in $\Gamma(H)$, where H is any complex separable Hilbert space, $(\{a(f), f \in H\}, \psi_0)$ is a SHS. Now suppose that $(\Omega, \mathcal{F}, P; \mathcal{H})$ is a Gaussian probability space as in Example 3. The Wiener-Segal isomorphism (see e.g. [13], pp. 66-7) establishes that there is a unitary isomorphism between $L^2(\Omega, \mathcal{F}, P; \mathbb{C})$ and $\Gamma(\mathcal{H})$ which maps finite products of Hermite polynomials to corresponding finite particle vectors. This establishes a strong equivalence between the SHS of Example 3 and the subsystem of classical type of $(\{a(f), f \in \mathcal{H}\}, \psi_0)$ which is generated by $\{a(f) + a^\dagger(f), f \in \mathcal{H}\}$.

Fix $\mathcal{I} = \mathbb{R}^+$. Following Cockroft and Hudson [8] and utilising the structure introduced above, we define a *(standard) quantum Brownian motion* to be a SHS for which

(i) $X(0) = 0$.

(ii) For all $s, t \in \mathbb{R}^+$, on $\mathcal{D}$,

$$[X(s), X(t)] = [X(s)^\dagger, X(t)^\dagger] = 0, [X(s), X(t)^\dagger] = (s \wedge t)I.$$

(iii) For all $n \in \mathbb{N}, T > 0$ and all partitions $\mathcal{P} = \{0 \leq t_0 < \cdots < t_n = T\}$ the operators $\{b_{j,T,\mathcal{P}}, 1 \leq j \leq n\}$ generate an i.i.d Gaussian PHS(n) with covariance I, where

$$b_{j,T,\mathcal{P}} := \frac{1}{\sqrt{t_j - t_{j-1}}}(X(t_j) - X(t_{j-1})).$$

Boson Fock Brownian motions are obtained by taking $H = L^2(\mathbb{R}^+)$, each $X(t) = a(1_{[0,t)})$ where $a(f)$ is the annihilation operator associated to $f \in H$, ψ_0 is the vacuum vector and $\mathcal{D}$ is the linear span of the exponential vectors.

Theorem 2.1 (Cockcroft-Hudson) *Any standard quantum Brownian motion is equivalent to the boson Fock Brownian motion.*

This result is proved in [8]. Note that Cockroft and Hudson define quantum Brownian motion in terms of families of pairs of self-adjoint operators. They also insert $-i$ in front of the right hand side of the non-trivial commutation relation in (i). This corresponds to defining the Fock space Brownian motion in terms of the canonical pair $((Q(t), P(t)), t \geq 0)$ where $Q(t) = a(1_{[0,t)}) + a^\dagger(1_{[0,t)})$ and $P(t) = i(a(1_{[0,t)}) - a^\dagger(1_{[0,t)}))$. [8] also present a more general version of Theorem 2.1 which allows constant multiples of the identity within the covariance structure in (iii). These lead to "non-Fock quantum Brownian motions" based on extremal universally invariant representations of the canonical commutation relations. We will comment further on these at the end of the next section.

3 The Lévy-Cielsielski Construction

We construct a quantum Brownian motion with index set $\mathcal{I} = [0, 1]$.

The *Haar system* is constructed as follows. The mother wavelet is

$$H(t) := \begin{cases} 1 & \text{if } 0 \leq t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The daughter wavelets are constructed by scaling and translation,

$$H_n(t) := 2^{\frac{j}{2}} H(2^j t - k), \quad n = 2^j + k, j \geq 0, 0 \leq k < 2^j.$$

If we define $H_0(t) := 1$, then $(H_n, n \in \mathbb{Z}_+)$ is a complete orthonormal basis for $L^2([0, 1])$.

The *Schauder system* is defined as follows. The mother wavelet is

$$\begin{aligned}\Delta(t) &:= 2\int_0^t H(u)du \\ &= \begin{cases} 2t & \text{if } 0 \leq t < \frac{1}{2} \\ 2(1-t) & \text{if } \frac{1}{2} \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases},\end{aligned}$$

and the daughter wavelets are $\Delta_n(t) := \Delta(2^j t - k)$ for $n = 2^j + k$ as above. If we define $\Delta_0(t) := t$, then $(\Delta_n, n \in \mathbb{Z}_+)$ is a Schauder basis (see e.g. Chapter 3 of [7]) for the Banach space $C_0[0,1]$ of continuous functions on $[0,1]$ which vanish at the origin, equipped with the usual supremum norm. Note in particular that

$$\sup_{n\in\mathbb{Z}_+}\sup_{t\in[0,1]}\Delta_n(t) = 1. \tag{3.2}$$

Furthermore, for each $n \in \mathbb{Z}_+$,

$$\Delta_n(t) = \frac{1}{\lambda_n}\int_0^t H_n(u)du, \tag{3.3}$$

where $\lambda_n := 2^{-\frac{j}{2}-1}$ for $n = 2^j + k$ as above, and $\lambda_0 := 1$.

We work in $\Gamma(l^2(\mathbb{Z}_+))$. For each $n \in \mathbb{Z}_+$, let $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots,)$, so $(e_n, n \in \mathbb{Z}_+)$ is an orthonormal basis for $l^2(\mathbb{Z}_+)$. Hence for each $g \in l^2(\mathbb{Z}_+), g = \sum_{n=0}^{\infty} g_n e_n$, where $g_n := \langle g, e_n\rangle$. We define $a_n := a(e_n)$ for each $n \in \mathbb{Z}_+$; then we have the canonical commutation relations:-

$$[a_m, a_n] = [a_n^{\dagger}, a_m^{\dagger}] = 0, [a_n, a_m^{\dagger}] = \delta_{mn},$$

for each $m, n \in \mathbb{Z}_+$. For all $f \in l^2(\mathbb{Z}_+), \psi(f)$ will denote the corresponding exponential vector. From now on we fix $H = \Gamma(l^2(\mathbb{Z}_+)), \mathcal{D}$ to be the linear span of the exponential vectors and ψ_0 to be the vacuum vector.

The following is well-known "Fock-law" and we omit the full proof (see e.g. [21], [17]).

Proposition 3.1 *1. $(\{a_n, n \in \mathbb{Z}_+\}, \psi_0)$ is a Gaussian SHS. In particular all the finite dimensional distributions are i.i.d. Gaussian.*

2.

$$||a_n\psi(g)|| = |g_n|e^{\frac{||g||^2}{2}},$$

for all $g \in l^2(\mathbb{Z}_+), n \in \mathbb{Z}_+$.

3.
$$||a_n^\dagger \psi(g)|| = (1+|g_n|^2)^{\frac{1}{2}} e^{\frac{||g||^2}{2}},$$
for all $g \in l^2(\mathbb{Z}_+), n \in \mathbb{Z}_+$.

Indeed, (1) is a straightforward extension of Example 1, (2) follows directly from the eigenrelation $a(f)\psi(g) = \langle f, g\rangle$, and (3) also follows from this via the commutation relations.

The proofs of the following are based (so far as is possible) on the corresponding construction for classical Brownian motion given in [23] pp. 36-9 and utilising Proposition 3.1 where appropriate.

Theorem 3.1 *The series*
$$Y(t)\psi := \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) a_n \psi \quad \text{and} \quad Y(t)^\dagger \psi := \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) a_n^\dagger \psi \tag{3.4}$$
converge uniformly for each $\psi \in \mathcal{D}$. *The linear operators* $Y(t)$ *and* $Y(t)^\dagger$ *which are so defined are closable with each* $Y(t)^\dagger \subseteq Y(t)^*$.

Furthermore the maps from $[0,1]$ *to* $\Gamma(l^2(\mathbb{Z}_+))$ *given by* $t \to Y(t)\psi$ *and* $t \to Y(t)^\dagger\psi$ *are continuous.*

Proof. It is sufficient to take $\psi = \psi(g)$ for some $g \in l^2(\mathbb{Z}_+)$. For any given $0 \le t \le 1$, we have $\Delta_n(t) = 0$ except for one n in each interval of the form $[2^j, 2^{j+1})$. We write each such n in the form $2^j + k_n$ where $0 \le k_n < 2^j$. Using proposition 3.1 (1), (3.2) and the Cauchy-Schwarz inequality, we have for sufficiently large $M \ge 2^J$ (say)

$$\begin{aligned}
\left|\left|\sum_{n=M}^{\infty} \lambda_n \Delta_n(t) a_n \psi\right|\right| &\le \sum_{n=M}^{\infty} \lambda_n \Delta_n(t) ||a_n \psi|| \\
&= \sum_{n=M}^{\infty} \lambda_n \Delta_n(t) |g_n| e^{\frac{1}{2}||g||^2} \\
&= \frac{1}{2} e^{\frac{1}{2}||g||^2} \sum_{j=J}^{\infty} \sum_{k=0}^{2^{j-1}} 2^{-\frac{j}{2}} \Delta_{2^j+k}(t) |g_{2^j+k}| \\
&= \frac{1}{2} e^{\frac{1}{2}||g||^2} \sum_{j=J}^{\infty} 2^{-\frac{j}{2}} |g_{2^j+k_n}| \\
&\le \frac{1}{2} e^{\frac{1}{2}||g||^2} \left(\sum_{j=J}^{\infty} 2^{-j}\right)^{\frac{1}{2}} \left(\sum_{j=J}^{\infty} |g_{2^j+k_n}|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{1}{2}||g||e^{\frac{1}{2}||g||^2}\left(\sum_{j=J}^{\infty}2^{-j}\right)^{\frac{1}{2}} \\
&\to 0 \quad \text{as } J \to \infty.
\end{aligned}
$$

Using similar arguments and proposition 3.1 (2), we obtain

$$
\begin{aligned}
\left|\left|\sum_{n=M}^{\infty}\lambda_n\Delta_n(t)a_n^{\dagger}\psi\right|\right| &\leq \frac{1}{2}e^{\frac{1}{2}||g||^2}\sum_{j=J}^{\infty}2^{-\frac{j}{2}}(1+|g_{2^j+k_n}|^2)^{\frac{1}{2}} \\
&\leq \frac{1}{2}e^{\frac{1}{2}||g||^2}\sum_{j=J}^{\infty}2^{-\frac{j}{2}}(1+|g_{2^j+k_n}|) \\
&\leq \frac{1}{2}e^{\frac{1}{2}||g||^2}\left[\sum_{j=J}^{\infty}2^{-\frac{j}{2}}+||g||\left(\sum_{j=J}^{\infty}2^{-j}\right)^{\frac{1}{2}}\right] \\
&\to 0 \quad \text{as } J \to \infty.
\end{aligned}
$$

The closure and mutual adjointness follow from the easily verified fact that for each $0 \leq t \leq 1, \phi_1, \phi_2 \in \mathcal{D}$,

$$\langle Y(t)^{\dagger}\phi_1, \phi_2\rangle = \langle \phi_1, Y(t)\phi_2\rangle,$$

and the continuity follows by the uniform convergence of each series on $[0,1]$. □

Theorem 3.2 $(\{Y(t), t \geq 0\}, \psi_0)$ *is a quantum Brownian motion.*

Proof. For each $0 \leq s, t \leq 1$ on the domain $\mathcal{D}$, using (3.3) and Parseval's identity, we have

$$
\begin{aligned}
[Y(s), Y(t)^{\dagger}] &= \sum_{m,n=0}^{\infty}\lambda_m\lambda_n\Delta_m(s)\Delta_n(t)[a_m, a_n^{\dagger}] \\
&= \sum_{m,n=0}^{\infty}\lambda_m\lambda_n\Delta_m(s)\Delta_n(t)\delta_{mn}I \\
&= \sum_{n=0}^{\infty}\lambda_n\Delta_n(s).\lambda_n\Delta_n(t)I \\
&= \sum_{n=0}^{\infty}\left(\int_0^s H_n(u)du\right).\left(\int_0^t H_n(u)du\right)I \\
&= \sum_{n=0}^{\infty}\langle 1_{[0,s]}, H_n\rangle\langle 1_{[0,t]}, H_n\rangle I \\
&= \langle 1_{[0,s]}, 1_{[0,t]}\rangle I = (s \wedge t)I,
\end{aligned}
$$

as required. The other two commutation relations are immediate.

To establish Gaussianity, let $\mathcal{P}$ be a partition of $[0,1]$ containing $n+1$ points and denote the associated operators $b_{j,1,\mathcal{P}}$ simply by $b_j (1 \leq j \leq n)$. It follows from the commutation relations that

$$[b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0, [b_j, b_k^\dagger] = \delta_{jk},$$

for each $1 \leq j, k \leq n$. The fact that these are independent Gaussians now follows from the argument given in the proof of theorem 1 in [8]. Alternatively, using proposition 20.15 in [21], for all $\alpha_j \in \mathbb{C} (1 \leq j \leq n)$

$$\begin{aligned}
& \left\langle \psi_0, \exp\left\{ \left(\sum_{j=1}^{n} (\alpha_j b_j^\dagger - \bar{\alpha_j} b_j) \right)^c \right\} \psi_0 \right\rangle \\
= & \left\langle \psi_0, \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} |\alpha_j|^2 \right\} \exp\left\{ \sum_{j=1}^{n} \alpha_j b_j^\dagger \right\} \exp\left\{ \sum_{j=1}^{n} \bar{\alpha_j} b_j \right\} \psi_0 \right\rangle \\
= & \exp\left\{ -\frac{1}{2} \sum_{j=1}^{n} |\alpha_j|^2 \right\}. \qquad \square
\end{aligned}$$

We have used wavelets to construct a quantum Brownian motion on $[0,1]$. To extend the index set to the whole of $\mathbb{R}^+$ we work in the countable (or incomplete) tensor product of an infinite number of copies of $\Gamma(l^2(\mathbb{N})$ with respect to the stabilising sequence comprising the vacuum vector in each space (see e.g. [21] p.95, [24]). We construct a countable number of i.i.d copies of quantum Brownian motion on $[0,1]$ via the prescription

$$Y^{(n)}(t) = I \otimes \cdots \otimes I \otimes \overset{(n)}{Y(t)} \otimes I \otimes \cdots$$

with domain the ampliation of $\mathcal{D}$. We then define quantum Brownian motion on $\mathbb{R}^+$ as follows. We take ψ_0 to be the infinite tensor product of vacuum vectors, the domain to be the incomplete algebraic tensor product of an infinite number of copies of $\mathcal{D}$ and the required operators $(A(t), t \geq 0)$ are given by

$$A(t) = \sum_{k=1}^{n} Y^{(k)}(1) + Y^{(n+1)}(t-n),$$

whenever $t \in (n, n+1]$ (c.f. [23],p. 40 for the classical case).

To establish the Lévy-Cielsieski construction for quantum Brownian motion of variance $\sigma^2 > 1$, we follow [11] and work in $\Gamma(l^2(\mathbb{Z}_+)) \otimes \Gamma(\overline{l^2(\mathbb{Z}_+)})$, where $\bar{\cdot}$ here denotes duality. In place of each $\psi(f)$ use $\psi(f) \otimes \psi(\overline{f})$ and in place of a_n use $\lambda a(e_n) \otimes I + \mu I \otimes a^{\dagger}(\overline{e_n})$, where $\lambda^2 = \frac{1}{2}(1+\sigma^2)$ and $\mu^2 = \frac{1}{2}(\sigma^2 - 1)$.

To construct fermion Brownian motion, we work in fermion (antisymmetric) Fock space $\Gamma_-(l^2(\mathbb{Z}_+ +))$. We replace each a_n by the corresponding fermion operator f_n which satisfies the canonical anti-commutation relations:

$$\{f_m, f_n\} = \{f_m^{\dagger}, f_n^{\dagger}\} = 0, \{f_m, f_n^{\dagger}\} = \delta_{mn}.$$

Since we have the estimates $||f_n|| = ||f_n^{\dagger}|| = 1$, we obtain uniform convergence of the series

$$F(t) := \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) f_n \quad \text{and} \quad F(t)^{\dagger} := \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) a_n^{\dagger},$$

in the norm topology on $B(\Gamma_-(l^2(\mathbb{Z}_+ +))$. Using techniques developed in [2], it is easily verified that $(F(t), t \geq 0), \psi_0)$ is a fermion Brownian motion, where ψ_0 is the vacuum vector in $\Gamma_-(l^2(\mathbb{Z}_+ +))$.

Bozejko and Speicher [6] consider a generalised Brownian motion based on the commutation relation

$$c(f)c(g)^{\dagger} - \mu c(g)^{\dagger} c(f) = \langle f, g \rangle I,$$

where $-1 \leq \mu \leq 1$, realised in a full Fock space. The special cases $\mu = -1, 0, 1$ give rise to fermionic, free (see also [4]) and bosonic Brownian motions (respectively). For $-1 \leq \mu < 1$, we have

$$||c(f)|| = \frac{1}{1-\mu} ||f||| \quad (0 \leq \mu < 1) \quad , \quad ||c(f)|| = ||f|| \quad (-1 \leq \mu \leq 0),$$

thus we have uniform convergence in the norm topology on bounded operators of the corresponding wavelet expansion.

As annihilation and creation operators in monotone Fock space are bounded, similar remarks to those above apply to the Lévy-Cielsieski construction of Muraki's monotone Brownian motion [20].

4 Quantum Brownian Bridges

We return to boson probability.

We define a *quantum Brownian bridge* to be a Gaussian SHS $(U(t), t \in [0,1]), \psi_0)$ for which

(i) $U(0) = U(1) = 0$.

(ii) For all $s, t \in [0,1]$, on $\mathcal{D}$,

$$[U(s), U(t)] = [U(s)^{\dagger}, U(t)^{\dagger}] = 0, [U(s), U(t)^{\dagger}] = (s \wedge t)[1 - (s \vee t)]I.$$

To construct a quantum Brownian bridge, let $(X(t), t \in [0,1]), \psi_0)$ be a quantum Brownian motion. Define

$$U(t) := X(t) - tX(1),$$

for each $t \in [0,1]$. This is a quantum Brownian bridge. Indeed (i) and (ii) are both trivial and Gaussianity follows from writing each $U(t) = (1 - t)X(t) - t(X(1) - X(t))$ and observing that $\{X(u), 0 \leq u \leq t), \psi_0\}$ and $\{(X(1) - X(u), t \leq u \leq 1), \psi_0\}$ are independent Gaussian SHSs. From the Cockroft-Hudson theorem 2.1, it follows that all quantum Brownian bridges are unitarily equivalent to that obtained from the Fock Brownian motion in this manner.

Using the Wiener-Segal duality transformation (see e.g. [21]), we see that for each $\theta \in [0, 2\pi), e^{i\theta}U(t) + e^{-i\theta}U(t)^{\dagger}$ is unitarily equivalent to a classical Brownian bridge in Wiener space. The cases $\theta = 0$ and $\theta = \frac{\pi}{2}$ are naturally associated to canonical position and momentum field operators.

The following results are easily established analogues of well-known classical results which can be found in e.g. [22].

1. If $(\{U(t), t \in [0,1]\}, \psi_0)$ is a quantum Brownian bridge then so is $(\{U(1-t), t \in [0,1]\}, \psi_0)$.

2. There is a one-to one correspondence between quantum Brownian motions $(\{A(t), t \in [0,1]\}, \psi_0)$ and quantum Brownian bridges $(\{U(t), t \in [0,1]\}, \psi_0)$ on the same space given by

$$A(t) \to (t+1)U\left(\frac{t}{t+1}\right), \quad \text{for all } t \in \mathbb{R}^+,$$

$$U(t) \to (1-t)A\left(\frac{t}{1-t}\right), \quad \text{for all } t \in [0,1).$$

Returning to the Lévy-Cielsieski expansions of theorems 3.1 and 3.2, it follows just as in the classical case ([23]) that $(\{V(t), t \in [0,1]\}, \psi_0)$ is a quantum Brownian bridge, where each

$$V(t) := \sum_{n=1}^{\infty} \lambda_n \Delta_n(t) a_n.$$

To see this, observe that since each $\Delta_n(1) = 0$ for each $n \in \mathbb{N}, Y^{\#}(1) = a_0^{\#}$, hence

$$V(t)^{\#} = Y(t)^{\#} - \Delta_0(t)a_0^{\#} = Y(t)^{\#} - tY(1)^{\#}.$$

The following is a quantum version of a well-known classical example of a Brownian bridge (see e.g. [10], theorem 1.10), although the method of the proof is completely different. We work in $\Gamma(L^2(\mathbb{R}^+))$.

Theorem 4.1 *($\{U(t), t \in [0,1]\}, \psi_0$) is a quantum Brownian bridge, where for each* $0 \leq t < 1$

$$U(t) := (1-t)\int_0^t \frac{dA(u)}{1-u}.$$

Proof. Gaussianity follows from the discussion at the end of section 2 and it is sufficient to verify the non-trivial commutation relation. For $0 \leq s \leq t < 1$, we use

$$U(t)^{\dagger} = (1-t)\int_0^s \frac{dA^{\dagger}(u)}{1-u} + (1-t)\int_s^t \frac{dA^{\dagger}(u)}{1-u}.$$

Then applying (2.1) and the canonical commutation relations:

$$\begin{aligned} [U(s), U(t)^{\dagger}] &= (1-s)(1-t)\left[\int_0^s \frac{dA(u)}{1-u}, \int_0^s \frac{dA^{\dagger}(u)}{1-u}\right] \\ &= (1-s)(1-t)\int_0^s \frac{du}{(1-u)^2} I \\ &= s(1-t)I. \qquad \square \end{aligned}$$

A straightforward application of the quantum Itô formula (as in e.g. [21], [17]) applied to the result of theorem 4.1 yields the quantum stochastic differential equation representation of a quantum Brownian bridge:

$$dU(t) = dA(t) - \frac{1}{1-t}U(t)dt,$$

for all $0 \leq t < 1$.

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